Logics of variable inclusion

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Abstract

A variety of algebras $\mathcal{K}$ is called *strongly irregular* whenever it satisfies an identity of the kind $f(x, y) \approx x$, where $f(x, y)$ is any term of the language in which x and y really occur. On the other hand, an identity $\varphi \approx \psi$ is said to be *regular* provided that exactly the same variables occur in φ and ψ. A variety is *regular*, when it is defined by regular identities. Examples of (strongly) irregular varieties abound in logic, since every variety with a lattice reduct is (strongly) irregular as witnessed by the term $f(x, y) \coloneqq x \wedge (y \vee x)$. The algebraic study of regular varieties traces back to the pioneering work of Płonka [7], who introduced a class-operator $\mathcal{P}_l(\cdot)$ nowadays called *Płonka sums*, and used it to prove that any regular variety $\mathcal{K}$ can be represented as Płonka sums of a suitable strongly irregular variety $\mathcal{V}$, in symbols $\mathcal{P}_l(\mathcal{V}) = \mathcal{K}$. In this case $\mathcal{K}$ is called the *regularization* of $\mathcal{V}$.

Over the years, regular varieties have been studied in depth from a purely algebraic perspective, see for example [1, 8, 5, 6, 4]. However, the recent discovery [2] that the regularization of the variety of Boolean algebras is the algebraic semantics of Paraconsistent Weak Kleene logic, i.e. a particular three-valued Kleene-like logic, showed that the notion of regularization could find interesting applications in logic as well. Building on this observation, in this contribution we develop a new notion of *logic-based* regularization, which on the one hand extends and subsumes the known algebraic one, and on the other hand explains on general grounds the relations between classical logic and Paraconsistent Weak Kleene logic discovered in [2]. Our investigation is carried on in the framework of abstract algebraic logic [3]. More in detail, given a logic $\vdash$, it is always possible to define a new logic $\vdash_r$, called *the logic of variable inclusion of* $\vdash$, according with the following clause:

$$\Gamma \vdash_r \varphi \iff \text{there is } \Delta \subseteq \Gamma \text{ s.t. } \mathrm{var}(\Delta) \subseteq \mathrm{var}(\varphi) \text{ and } \Delta \vdash \varphi.$$

In order to see why logics of variable inclusion can be considered as the logical counterpart of regular varieties we need to extend first the construction of Płonka sums to logical matrices. A *direct system* of logical matrices is a triple $X = \langle I, \{\mathbf{A}_i, F_i\}_{i \in I}, \varphi_{ij} \rangle$ where, I is a semilattice of indexes, $\langle \mathbf{A}_i, F_i \rangle_{i \in I}$ is a family of logical matrices, and $\varphi_{ij} : \mathbf{A}_i \to \mathbf{A}_j$ is a homomorphism such that $\varphi_{ij}(F_i) \subseteq F_j$ for every $i \leq j$. The Płonka sum over a direct system X is the logical matrix $\mathcal{P}_l(X) \coloneqq \langle \mathcal{P}_l(\mathbf{A}_i), \bigcup_{i \in I} F_i \rangle$. Now, recall that every logic $\vdash$ can be naturally associated with a class of logical matrices, usually called the *reduced models* of $\vdash$. As first result, we prove that any logic of variable inclusion $\vdash^r$ is complete w.r.t. all the Płonka sums over a precise class of direct systems of models of $\vdash$. That is, syntactically defined logics of variable inclusion actually correspond to logics semantically defined via Płonka sums. From this starting point, we investigate logico-algebraic properties of $\vdash_r$ such as its location in the Leibniz and Frege hierarchy (both as a logic and as a Gentzen system), the structure of its Suszko reduced models and, moreover, we describe a general method to construct a complete Hilbert style calculus for $\vdash^r$ starting from a Hilbert calculus for $\vdash$.

References

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