Remarks on the Exponential Rules in Linear Logic

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We work in linear logic with weakening and consider the standard exponential rules

$$\frac{\Gamma, !A, !A \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta}\ (!c) \quad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta}\ (!d) \quad \frac{!\Gamma \Rightarrow A}{!\Gamma \Rightarrow !A}\ (!R)$$

The rules (!c) and (!d) say that contraction is applicable to formulas prefixed with ! ('bang'). In the intuitive interpretation of formulas as resources, $!A$ thus denotes a resource which can be used arbitrarily often.
In the same spirit, the *soundness* of the rule (!R) is sometimes argued for as follows: If we can obtain the resource A from resources $!\Gamma$, then we can repeat this 'process' to obtain arbitrarily many A's, since we never run out of the unbounded resources in $!\Gamma$.
Since the resource interpretation of linear logic is not formalized, no direct argument for the *completeness* of the rule (!R) can be made. In fact, some derivations seem to conflict with the interpretation of $!A$ as 'arbitrarily many'. For example, we can prove

$$C, !(C \to A \otimes C) \Rightarrow \underbrace{A \otimes \ldots \otimes A}_{n}$$

for every n, but we cannot prove

$$C, !(C \to A \otimes C) \Rightarrow !A$$

It is thus not clear which notion of 'arbitrarily many' is captured by the rule (!R). We try to gain some insight into this problem by comparing (!R) to a naive infinitary rule of the form

$$\frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow A \otimes A \qquad \Gamma \Rightarrow A \otimes A \otimes A \qquad \ldots}{\Gamma \Rightarrow !A}$$

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