Syntactic decidability and complexity upper bound for the logic of Bunched Implication BI

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The logic of Bunched Implication BI

- The logic of bunched implication BI introduced by (O'Hearn and Pym, 1999) to reason about resource composition
- Distinct from Linear logic. Two implications $-\!\!*$ and $\rightarrow$.
- The lattice connectives distribute: $A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$.
- There are the multiplicative connectives $\otimes$, $-\!\!*$ and unit $\mathbf{1}$ in addition to the intuitionistic connectives $\wedge$, $\vee$ and $\rightarrow$.
- BI was shown to be decidable with finite model property using resource tableaux (Galmiche, Méry, Pym 2005)
- In addition to the resource semantics, BI also has an algebraic semantics. Heyting algebras carrying an additional commutative associative structure $(\otimes, -\!\!*, \mathbf{1})$ such that $\mathbf{1}$ is the unit and for every x, y, z: $x \otimes y \leq z$ iff $x \leq y -\!\!* z$.
- Many different proof systems for BI. Arguably the simplest is the bunched (sequent) calculus **LBI**.
- (Ciabattoni and R., 2017) obtain structural rule extensions of **LBI** with cut-elimination for a large class of extensions of BI.

The question we consider here is a syntactic proof of decidability...

What is a syntactic proof and why do we want one?

- A syntactic proof (as opposed to a semantic one) argues solely on the logical syntax (via its proof calculus). No reference is made to the models of the logic.
- The argument can thus be checked without any knowledge of semantics.
- Complexity upper bounds can typically be obtained from an explicit bound on derivations in the calculus.
- The argument can be generalised to suitable extensions of the base logic
- Such proofs are technically interesting and may generalise/inspire proofs for other substructural logics
- The proof-theory of BI is elegant and deceptively simple so there is an added motivation to understand what is going on

Let us begin by introducing the proof calculus for BI. . .

- A formula of BI has the grammar

$$A := p \in \mathcal{V} \mid \top \mid \bot \mid \mathbf{1} \mid (A \vee A) \mid (A \wedge A) \mid (A \to A) \mid (A \otimes A) \mid (A \mathrel{-\!\!*} A)$$

- A bunch has the grammar: $X := A \in \mathrm{Fm} \mid \varnothing_a \mid \varnothing_m \mid (X, X) \mid (X; X)$
- $\varnothing_a$ is additive structure constant. $\varnothing_m$ is multiplicative structure constant.
- A bunched sequent is $X \Rightarrow B$ for a bunch X and formula B.
- Proof theory: combine the calculus for multiplicative intuitionistic linear logic

$$\varnothing_m \Rightarrow \mathbf{1} \qquad \frac{\Gamma[\varnothing_m, X] \Rightarrow A}{\Gamma[X] \Rightarrow A}\ (\varnothing_m\mathrm{E/I}) \qquad \frac{\Gamma[\varnothing_m] \Rightarrow A}{\Gamma[\mathbf{1}] \Rightarrow A}\ (\mathbf{1}\mathrm{I}) \qquad \frac{\Gamma[(X, Y), Z] \Rightarrow A}{\Gamma[X, (Y, Z)] \Rightarrow A}\ (\text{as-c})$$

$$\frac{\Gamma[X, Y] \Rightarrow A}{\Gamma[Y, X] \Rightarrow A}\ (\text{ex-c}) \qquad \frac{\Gamma[X] \Rightarrow A}{\Gamma[\varnothing_m, X] \Rightarrow A}\ (\varnothing_m\mathrm{I}) \qquad \frac{Y \Rightarrow C \qquad \Gamma[D] \Rightarrow A}{\Gamma[Y, C \mathrel{-\!\!*} D] \Rightarrow A}\ (\mathrel{-\!\!*}\mathrm{I})$$

$$\frac{X, C \Rightarrow D}{X \Rightarrow C \mathrel{-\!\!*} D}\ (\mathrel{-\!\!*}r) \qquad \frac{\Gamma[C, D] \Rightarrow A}{\Gamma[C \otimes D] \Rightarrow A}\ (\otimes\mathrm{I}) \qquad \frac{X \Rightarrow C \qquad Y \Rightarrow D}{X, Y \Rightarrow C \otimes D}\ (\otimes\mathrm{r})$$

- Comma is the structural connective for $\otimes$
- Comma is associative and commutative. Its unit is $\varnothing_m$

- . . . with the calculus for intuitionistic logic. . .

$$p \Rightarrow p \qquad \Gamma[\bot] \Rightarrow C \qquad X \Rightarrow \top$$

$$\frac{\Gamma[X; X] \Rightarrow A}{\Gamma[X] \Rightarrow A}\ (\text{ctr}) \qquad \frac{\Gamma[\varnothing_a] \Rightarrow A}{\Gamma[\top] \Rightarrow A}\ (\top\text{l}) \qquad \frac{\Gamma[X] \Rightarrow A}{\Gamma[X; Y] \Rightarrow A}\ (\text{w})$$

$$\frac{\Gamma[(X; Y); Z] \Rightarrow A}{\Gamma[X; (Y; Z)] \Rightarrow A}\ (\text{as-sc}) \qquad \frac{\Gamma[X; Y] \Rightarrow A}{\Gamma[Y; X] \Rightarrow A}\ (\text{ex-sc}) \qquad \frac{\Gamma[\varnothing_a; X] \Rightarrow A}{\Gamma[X] \Rightarrow A}\ (\varnothing_a\text{E})$$

$$\frac{Y \Rightarrow C \qquad \Gamma[D] \Rightarrow A}{\Gamma[Y; C \to D] \Rightarrow A} \to\text{l} \qquad \frac{\Gamma[C] \Rightarrow A \qquad \Gamma[D] \Rightarrow A}{\Gamma[C \vee D] \Rightarrow A}\ (\vee\text{l}) \qquad \frac{X; C \Rightarrow D}{X \Rightarrow C \to D}\ (\to r)$$

$$\frac{\Gamma[C_i] \Rightarrow A}{\Gamma[C_1 \wedge C_2] \Rightarrow A}\ (\wedge\text{l}) \qquad \frac{X \Rightarrow C \qquad X \Rightarrow D}{X \Rightarrow C \wedge D}\ (\wedge\text{r}) \qquad \frac{\Gamma \Rightarrow C_i}{\Gamma \Rightarrow C_1 \vee C_2}\ (\vee\text{r})$$

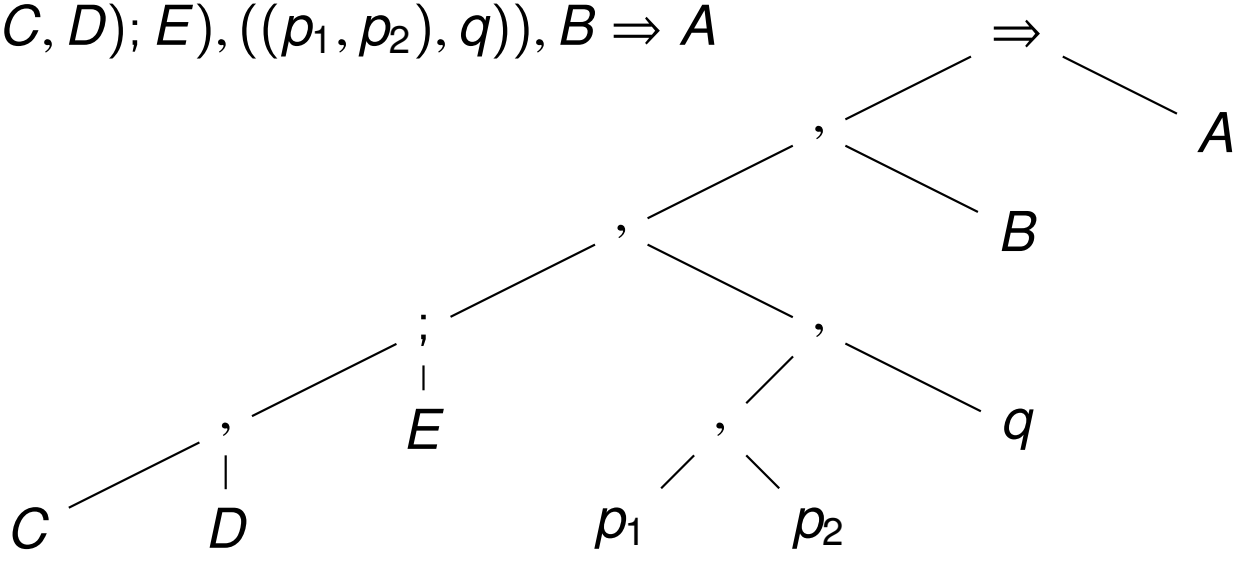
- Semicolon is the structural connective for $\wedge$
- Semicolon has associative, commutative, contraction and weakening properties, its unit is $\varnothing_a$
- The BI calculus has the subformula property: the formulae that appear in a derivation of $\mathbf{1} \Rightarrow F$ are subformulae of F

Proving decidability syntactically via the proof calculus

- A backward proof (bp) from $\mathbf{1} \Rightarrow F$ is obtained by successive application of rules backwards i.e. from conclusion to premise(s).
- A bp can be viewed as a proof attempt/partial proof. A derivation is a special case of a bp
- The usual plan (expressible in different ways) is to show that it is possible to effectively construct enough bps so that: $\mathbf{1} \Rightarrow F$ is derivable iff there is a derivation of $\mathbf{1} \Rightarrow F$ among the constructed bps.
- If the number of different sequents that can occur in a bp from $\mathbf{1} \Rightarrow F$ is calculable then any bp containing a branch of length greater than this number must repeat a sequent along this branch.
- Since we can restrict our search to bps such that no branch contains a repeated sequent (this strategy is complete because if $\mathbf{1} \Rightarrow F$ has a derivation, then it has a derivation such that no branch in it contains a repeat):
- If the set of bps whose branches do not contain repeats is effectively constructible and if checking if a bp is a derivation is effective, then we can effectively check if any of the bps is a derivation. If not, then $\mathbf{1} \Rightarrow F$ is not derivable.

- Every bunched sequent is built from subformulae sf(F) of F (by the subformula property) using the structural connectives.
- Helpful to think of bunched sequent via its grammar (syntax) tree

$(((C, D); E), ((p_1, p_2), q)), B \Rightarrow A$



- To bound the size of a sequent (and hence bound the number of different sequents), if suffices to bound the size of the bunch.
- Bounding comma depth is the focus here because it is the main obstacle
- Define the (comma) depth of the bunch as the maximum number of commas along a branch of the bunch.
- In the above example, the depth of the bunch is 4.

Kaminski-Francez bound & the gap in their argument

- Let $\#_{\mathsf{NL}}(F)$ be the number of Lambek connectives in F, i.e. $\otimes$ or $\multimap$.
- (Kaminski and Francez, 2016) claim:

Lemma

Let $\Gamma_0 \Rightarrow A_0$ *be a sequent that occurs in the* ***LBI****-derivation of a sequent* $\mathbf{1} \Rightarrow F$ *(*$F \in \mathsf{Fm}$*). Then* $\mathsf{d}(\Gamma_0) \leq \#_{NL}(F)$*.*

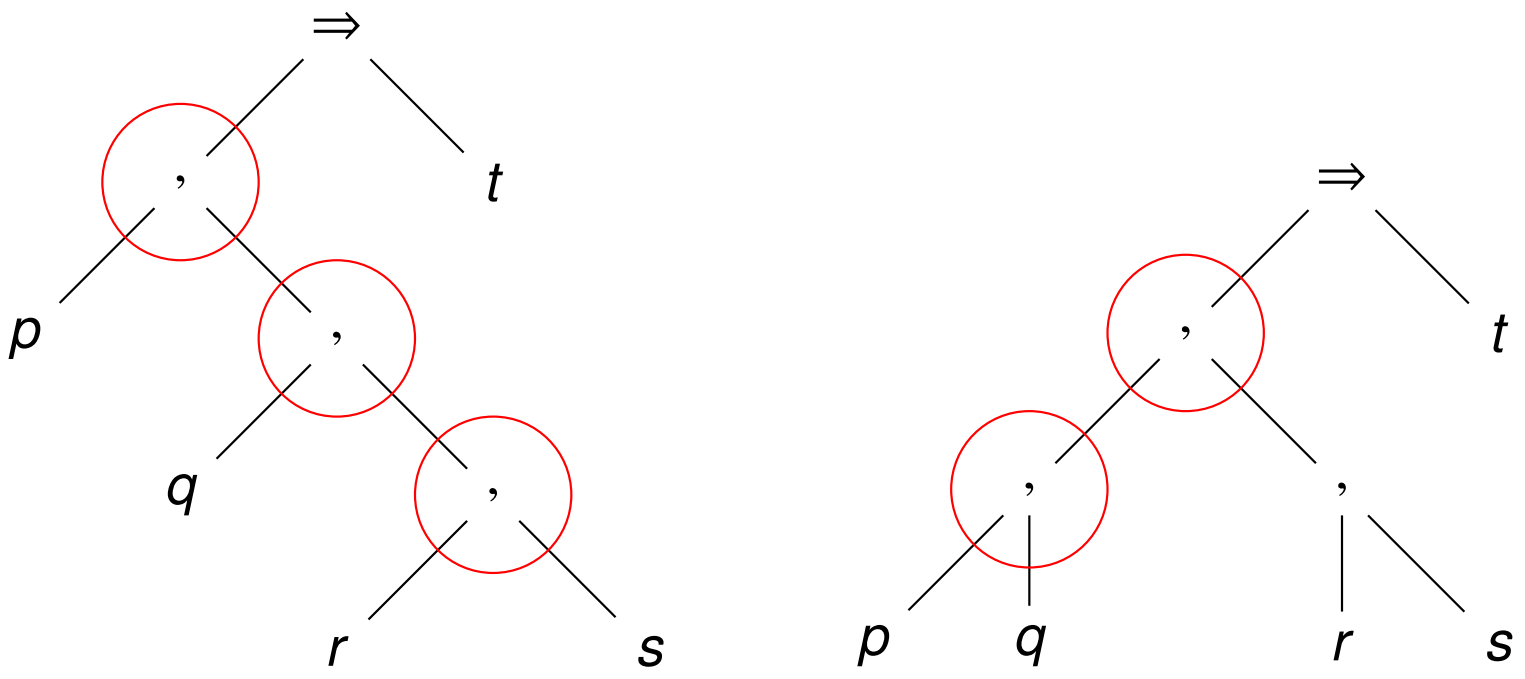
> *The depth of a bunch can be decreased [from premises to conclusion] only by the rules* $(\otimes l)$ *or* $(\multimap r)$*, each introducing the corresponding Lambek connective. Since the derivation is cut-free, all this [sic] connectives must occur in* F*.*

However, the above argument has the following serious issues:

1. It does not rule out the possibility that multiple commas along the branch correspond to the same Lambek connective occurrence in F. I.e. establishing that the map from the commas along a branch to the Lambek connectives in F is 1-to-1 is crucial (recall: contraction rule can 'merge' multiple commas in its premise to a single comma in its conclusion)

2. The statement that the depth of a bunch can be decreased [*from premises to conclusion*] only by the rules ($\otimes$l) or ($-\!\!*$r) is incorrect; the associative rules (as-c) and (as-sc) can also decrease the depth from premise to conclusion

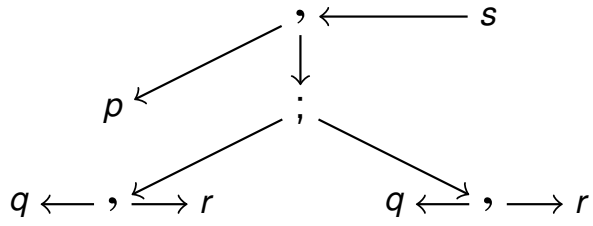
E.g. premise $p, (q, (r, s)) \Rightarrow t$ and conclusion $(p, q), (r, s) \Rightarrow t$:

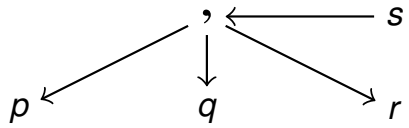


Gap in (Galatos and Jipsen, 2017) argument

- More recently, (Galatos and Jipsen, 2017) introduce Generalised BI algebras as a semantics for BI. A syntactic calculus is also obtained.
- From the bunched sequent grammar tree they define a directed graph (adding additional edges e.g. from a negative polarity $-\!\!*$ to a negative comma directly above it, negative $-\!\!*$ to a positive $-\!\!*$ above it, . . .)
- Crucial claim: the multiplicative length—the maximum (over all directed paths in the directed graph) number of negative polarity $\otimes$ and commas and positive polarity $-\!\!*$—is non-increasing from conclusion to premise(s).
- Comma treated as a *n*-ary connective rather than binary. Otherwise associativity rule for comma already violates the claim since e.g. premise $((p, q), r), s \Rightarrow t$ would have greater multiplicative length than $(p, q), (r, s) \Rightarrow t$
- However in the rule instance below left (premise and conclusion are drawn below centre and right): the multiplicative length for the premise is 2 and for the conclusion it is 1 so the claim on multiplicative length is invalidated

$$\frac{p, ((q, r); (q, r)) \Rightarrow s}{p, (q, r) \Rightarrow s}\ (\text{ctr})$$





A proof-theoretic argument establishing the Kaminski-Francez bound, and hence decidability

In the remainder of this talk I will sketch a proof of the Kaminski-Francez bound:

Lemma (bounding comma depth)

Let $\Gamma_0 \Rightarrow A_0$ *be a sequent that occurs in the* ***LBI****-derivation of a sequent* $\mathbf{1} \Rightarrow F$ *(*$F \in \mathrm{Fm}$*). Then* $\mathrm{d}(\Gamma_0) \leq \#_{NL}(F)$ *(number of Lambek connectives in* F*).*

1. Assign an index from $1, \ldots, N$ to every Lambek connective in F

$$((((r \overset{1}{\twoheadrightarrow} s) \rightarrow t) \overset{2}{\otimes} ((p_1 \wedge p_2) \overset{3}{\otimes} q)) \rightarrow l) \overset{4}{\twoheadrightarrow} m$$

2. Extend to a labelling of every Lambek connective and comma in the derivation of $\mathbf{1} \Rightarrow F$. Every label is a sequence $j, (1,2)^*$ $(1 \leq j \leq N)$.

► To extend the labelling from the endsequent to the rest of the derivation, deduce in an obvious way (for every rule excepting $(\varnothing_m \mathrm{E})$ I explain why later), the labelling of the premise(s) from the labelling of the conclusion

Labelling for the additive rules of BI:

$$p \Rightarrow p \qquad \Gamma[\bot] \Rightarrow C \qquad X \Rightarrow \top$$

$$\frac{\Gamma[\overset{\#1}{X}; \overset{\#2}{X}] \Rightarrow A}{\Gamma[X] \Rightarrow A}\ (\text{ctr}) \qquad \frac{\Gamma[\varnothing_a] \Rightarrow A}{\Gamma[\top] \Rightarrow A}\ (\top\text{l}) \qquad \frac{\Gamma[X] \Rightarrow A}{\Gamma[X; Y] \Rightarrow A}\ (\text{w})$$

$$\frac{\Gamma[(X; Y); Z] \Rightarrow A}{\Gamma[X; (Y; Z)] \Rightarrow A}\ (\text{as-sc}) \qquad \frac{\Gamma[X; Y] \Rightarrow A}{\Gamma[Y; X] \Rightarrow A}\ (\text{ex-sc}) \qquad \frac{\Gamma[\varnothing_a; X] \Rightarrow A}{\Gamma[X] \Rightarrow A}\ (\varnothing_a\text{E})$$

$$\frac{Y \Rightarrow C \qquad \Gamma[D] \Rightarrow A}{\Gamma[Y; C \to D] \Rightarrow A} \to\text{l} \qquad \frac{\Gamma[\overset{\#1}{C}] \Rightarrow A \qquad \Gamma[\overset{\#2}{D}] \Rightarrow A}{\Gamma[C \vee D] \Rightarrow A}\ (\vee\text{l}) \qquad \frac{X; C \Rightarrow D}{X \Rightarrow C \to D}\ (\to r)$$

$$\frac{\Gamma[C_i] \Rightarrow A}{\Gamma[C_1 \wedge C_2] \Rightarrow A}\ (\wedge\text{l}) \qquad \frac{\overset{\#1}{X} \Rightarrow C \qquad \overset{\#2}{X} \Rightarrow D}{X \Rightarrow C \wedge D}\ (\wedge\text{r}) \qquad \frac{\Gamma \Rightarrow C_i}{\Gamma \Rightarrow C_1 \vee C_2}\ (\vee\text{r})$$

- $\overset{\#1}{X}$ and $\overset{\#2}{X}$ denote the appending of “1" and “2" resp., to each label in X.
- E.g. if $X = (p \overset{3112}{\otimes} q)^{5}_{;}(r \overset{72}{\multimap} s)$ then $\overset{\#1}{X} = (p \overset{31121}{\otimes} q)^{51}_{;}(r \overset{721}{\multimap} s)$

Labelling for the multiplicative rules of BI:

$$\varnothing_m \Rightarrow \mathbf{1} \qquad \frac{\Gamma[\varnothing_m] \Rightarrow A}{\Gamma[\mathbf{1}] \Rightarrow A}\ (\mathbf{1}\mathrm{l}) \qquad \frac{\Gamma[(X \,;^{\sigma}\, Y) \,;^{\pi}\, Z] \Rightarrow A}{\Gamma[X \,;^{\sigma}\, (Y \,;^{\pi}\, Z)] \Rightarrow A}\ (\text{as-c})$$

$$\frac{\Gamma[X \,;^{\sigma}\, Y] \Rightarrow A}{\Gamma[Y \,;^{\sigma}\, X] \Rightarrow A}\ (\text{ex-c}) \qquad \frac{\Gamma[X] \Rightarrow A}{\Gamma[\varnothing_m \,;^{\sigma}\, X] \Rightarrow A}\ (\varnothing_m\mathrm{l}) \qquad \frac{Y \Rightarrow C \qquad \Gamma[D] \Rightarrow A}{\Gamma[Y \,;^{\sigma}\, C \mathrel{-\!\!*} D] \Rightarrow A}\ (\mathrel{-\!\!*}l)$$

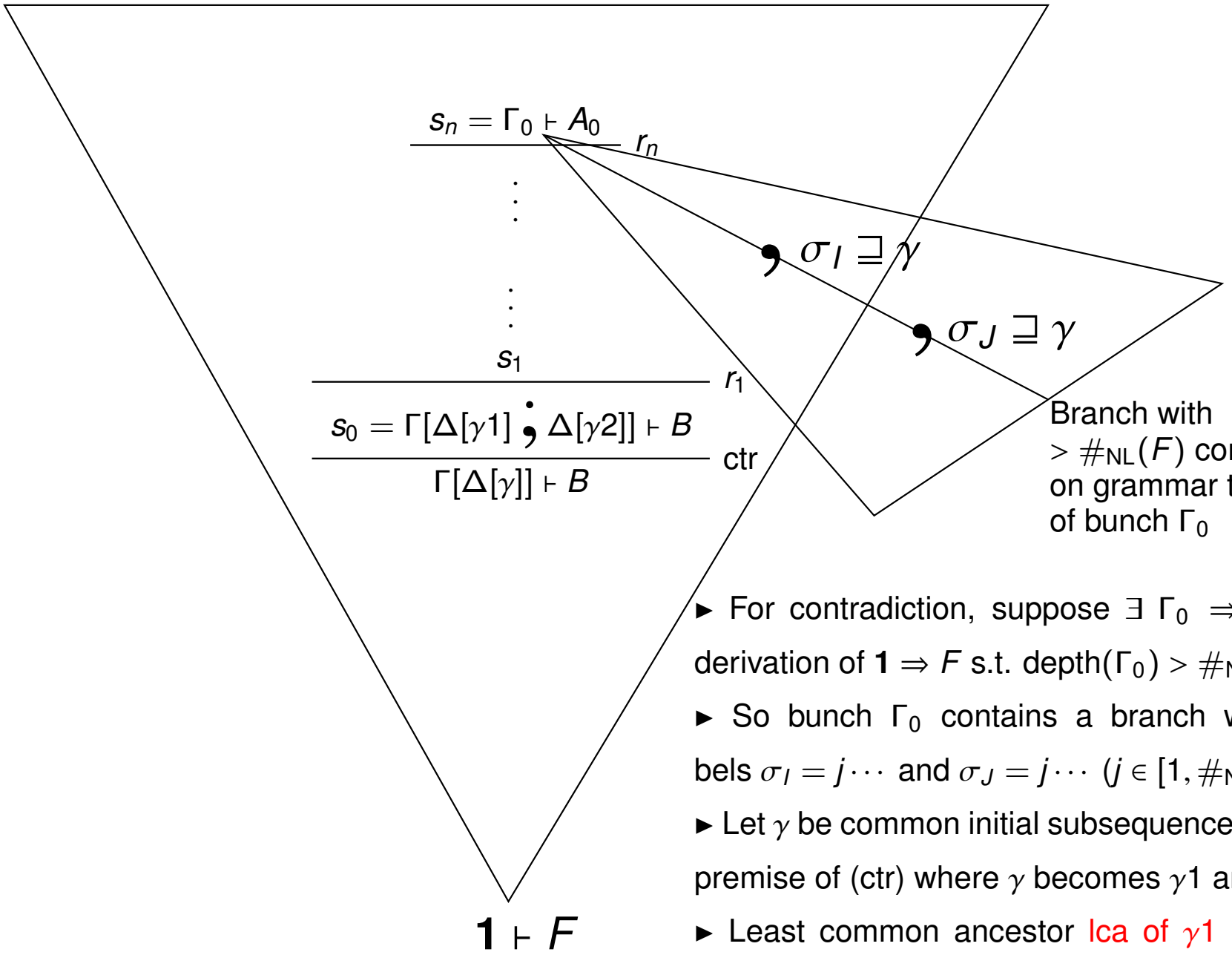
$$\frac{X \,;^{\sigma}\, C \Rightarrow D}{X \Rightarrow C \overset{\sigma}{\mathrel{-\!\!*}} D}\ (\mathrel{-\!\!*}r) \qquad \frac{\Gamma[C \,;^{\sigma}\, D] \Rightarrow A}{\Gamma[C \overset{\sigma}{\otimes} D] \Rightarrow A}\ (\otimes\mathrm{l}) \qquad \frac{X \Rightarrow C \qquad Y \Rightarrow D}{X \,;^{\sigma}\, Y \Rightarrow C \otimes D}\ (\otimes\mathrm{r})$$

$$\frac{X \Rightarrow C_i \qquad \varnothing_m \Rightarrow C_j}{X \Rightarrow C_1 \otimes C_2}\ (\otimes\mathrm{r})' \qquad \frac{\varnothing_m \Rightarrow C \qquad \Gamma[Y^{\sigma};D] \Rightarrow A}{\Gamma[Y^{\sigma};C \mathrel{-\!\!*} D] \Rightarrow A}\ (\mathrel{-\!\!*}\mathrm{l})'$$

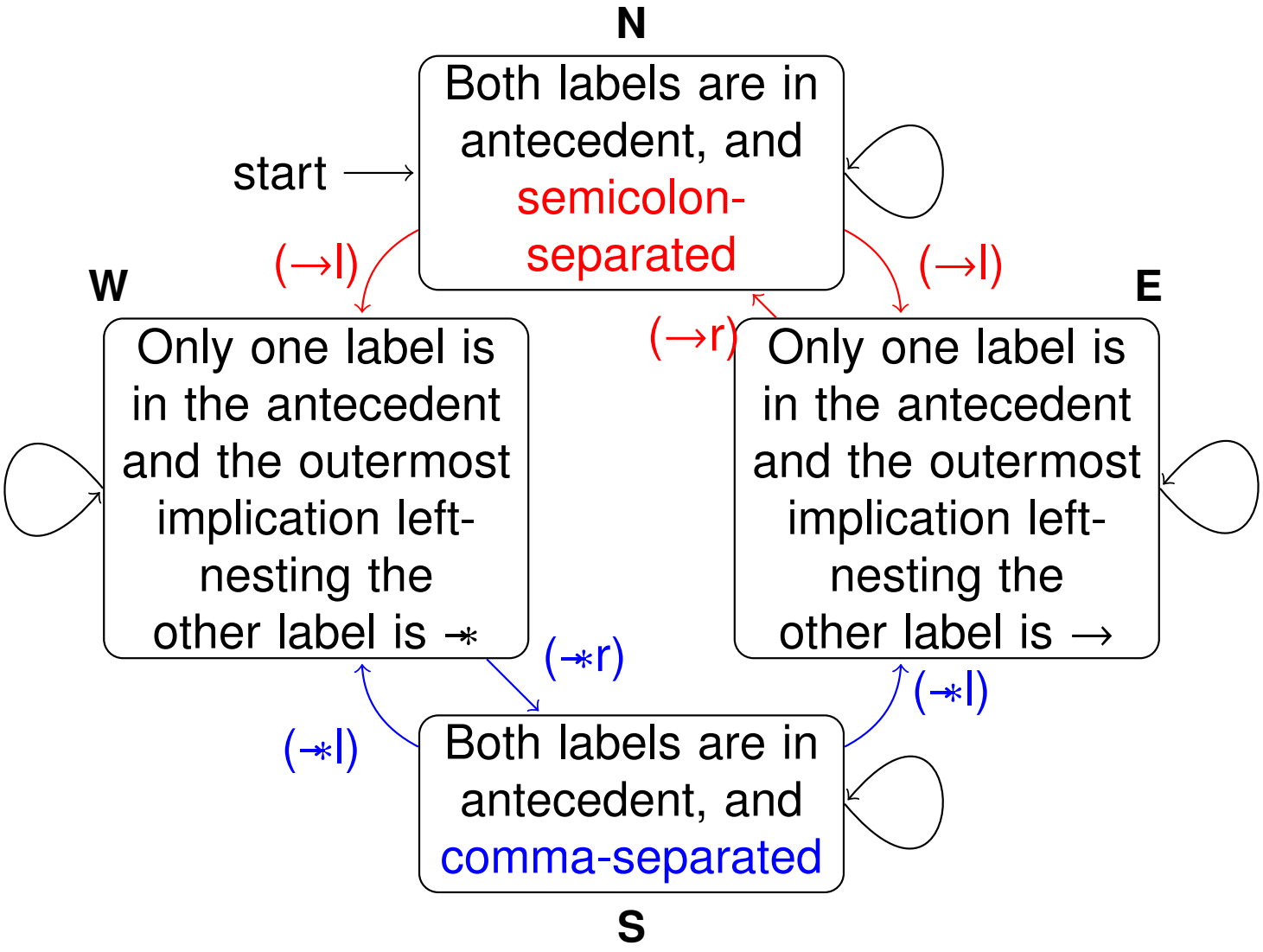
- Here are some examples.

$$\frac{X[((p \overset{\sigma 1}{\mathrel{-\!\!*}} (q \overset{\pi 1}{*} r)\,;s) \,;^{\delta 1}\, t)\,;((p \overset{\sigma 2}{\mathrel{-\!\!*}} (q \overset{\pi 2}{*} r)\,;s) \,;^{\delta 2}\, t)] \Rightarrow B}{X[(p \overset{\sigma}{\mathrel{-\!\!*}} (q \overset{\pi}{*} r)\,;s) \,;^{\delta}\, t] \Rightarrow B}\ (\text{ctr})$$

$$\frac{p \,;^{\sigma}\, q \Rightarrow r \qquad t\,;s \Rightarrow B}{t\,;((p \,;^{\sigma}\, q) \,;^{\pi}\, r \overset{\delta}{\mathrel{-\!\!*}} s) \Rightarrow B}\ (\mathrel{-\!\!*}\mathrm{l})$$



- For contradiction, suppose $\exists\ \Gamma_0 \Rightarrow A_0$ in derivation of $\mathbf{1} \Rightarrow F$ s.t. $\mathrm{depth}(\Gamma_0) > \#_{\mathsf{NL}}(F)$
- So bunch Γ_0 contains a branch with labels $\sigma_I = j \cdots$ and $\sigma_J = j \cdots$ ($j \in [1, \#_{\mathsf{NL}}(F)]$)
- Let γ be common initial subsequence and s_0 premise of (ctr) where γ becomes $\gamma 1$ and $\gamma 2$
- Least common ancestor lca of $\gamma 1$ and $\gamma 2$ in s_0: semicolon. lca of σ_I & σ_J in s_n: comma



- $s_0 \in \mathbf{N}$ and $s_n \in \mathbf{S}$. Every path from **N** to **S** is $\to y1\, y1 \ldots y_s y_s \twoheadrightarrow$ for $y_i \in \{\to, \twoheadrightarrow\}$

Eg. $N \to E \overbrace{\to N \to} E \overbrace{\to N \to} W \twoheadrightarrow S$

$N \to W \underbrace{\twoheadrightarrow S \twoheadrightarrow} E \underbrace{\to N \to} W \underbrace{\twoheadrightarrow S \twoheadrightarrow} W \twoheadrightarrow S$

Eg. $N \to E \overbrace{\to N \to} E \overbrace{\to N \to} W \twoheadrightarrow S$

$N \to W \underbrace{\twoheadrightarrow S \twoheadrightarrow} E \underbrace{\to N \to} W \underbrace{\twoheadrightarrow S \twoheadrightarrow} W \twoheadrightarrow S$

▶ Where do these implications come from? At s_0, each of $\gamma 1$ and $\gamma 2$ were left nested under some sequence $(\iota_1, \ldots, \iota_{2K})$ of implication connectives ($K \geq 1$)

$$\frac{s_0 = \Gamma[\Delta[\gamma 1]; \Delta[\gamma 2]] \vdash B}{\Gamma[\Delta[\gamma]] \vdash B}\ \text{ctr}$$

▶ At $s_n = \Gamma_0 \Rightarrow A_0$, all of the $4K$ implications—two copies of $(\iota_1, \ldots, \iota_{2K})$—have been removed

order of left-nesting imp removal:	1	2	3	4	5	...	$4K-2$	$4K-1$	$4K$
the connective that was removed:	$\to$	y_1	y_1	y_2	y_2	...	y_{2K-1}	y_{2K-1}	$\twoheadrightarrow$

▶ The total number of $\twoheadrightarrow$ in $y_1\, y_1\, y_2\, y_2 \ldots y_{2k-1}\, y_{2k-1}$ is necessarily even.

▶ Therefore the total number of $\twoheadrightarrow$ in $\to y1 y1 \ldots y_{2K-1} y_{2K-1} \twoheadrightarrow$ is odd.

▶ Since the $4K$ implications come from two copies of $(\iota_1, \ldots, \iota_{2K})$, the total number of $\twoheadrightarrow$ removed should be even. This is a contradiction.

Some proof details that were omitted for brevity

1. The proof uses the fact that no rule can create new commas in the premise. However, the rule ($\varnothing_m$E) does exactly that:

$$\frac{\Gamma[\varnothing_m, X] \Rightarrow A}{\Gamma[X] \Rightarrow A}\ (\varnothing_m\mathrm{E})$$

(Indeed, what label could we give the premise comma?)

   - The solution is to eliminate ($\varnothing_m$E) from the calculus. We show that this can be achieved if we add two new rules:

$$\frac{X \Rightarrow C_i \qquad \varnothing_m \Rightarrow C_j}{X \Rightarrow C_1 \otimes C_2}\ (\otimes\mathrm{r})' \qquad \frac{\varnothing_m \Rightarrow C \qquad \Gamma[D] \Rightarrow A}{\Gamma[C \mathrel{-\!\!*} D] \Rightarrow A}\ (\mathrel{-\!\!*}\mathrm{l})'$$

2. We need to bound semicolons, else sequent size could be unbounded.

   - After all, $\overbrace{(((p; p); p)\ldots); p}^{n+1 \text{ times}} \Rightarrow p$ has comma depth 0 for every n.
   - Nevertheless, there is a well-known technique (Curry's lemma) for controlling the branching width of a structural connective—like semicolon—that has weakening and contraction

1. Elimination of ($\varnothing_m$E) rule

- Consider the following instance $(\text{ctr})_{\varnothing_m}$ of the contraction rule.

$$\frac{\Gamma[(\varnothing_m; \varnothing_m)] \Rightarrow A}{\Gamma[\varnothing_m] \Rightarrow A}\ (\text{ctr})_{\varnothing_m}$$

- The height minus $(\text{ctr})_{\varnothing_m}$ of a derivation is the maximum number of successive rules in the derivation not counting instances of $(\text{ctr})_{\varnothing_m}$.
- Define $\mathcal{U} := \varnothing_m \mid \varnothing_a \mid (\mathcal{U}, \mathcal{U}) \mid (\mathcal{U}; \mathcal{U})$.

Lemma

*Let d be an **LBI**-derivation of the sequent s. For $U \in \mathcal{U}$:*

(I) *If s is $\Gamma[U] \Rightarrow A$ then $\Gamma[\varnothing_m] \Rightarrow A$ is **LBI**-derivable.*

(II) *If s is $\Gamma[U, M] \Rightarrow A$ or $\Gamma[M, U] \Rightarrow A$, then $\Gamma[M] \Rightarrow A$ is **LBI**-derivable.*

In each case, the new derivation has height minus $(\text{ctr})_{\varnothing_m}$ no greater than the original derivation.

- We prove (**??**) and (**??**) simultaneously, by induction on the height minus $(\text{ctr})_{\varnothing_m}$ of d. Consider the last rule that is not $(\text{ctr})_{\varnothing_m}$.
- Note: The proof of (**??**) makes use of $(\text{ctr})_{\varnothing_m}$ when the last rule is (as-sc).

2. Bounding semicolons

► Curry's lemma bounds the number of semicolon separated identical structures that need to be retained in a sequent (argument used by Gentzen)

Definition

A sequent is *k*-reduced ($k \geq 1$) if there is no sequence of (as-sc) and (ex-sc) rules that can be applied to it to obtain a sequent of the form

$$\Gamma[\overbrace{(\ldots((X;X);X);\ldots);X}^{k+1 \text{ occurrences of } X}] \Rightarrow A$$

► (Kaminski and Francez, 2016) and (Galatos and Jipsen, 2017) provide a proof of Curry's lemma for **LBI**

Theorem (Curry's lemma for LBI)

Every 3*-reduction of a* ***LBI****-derivable sequent has a* 3*-reduced* ***LBI****-derivation.*

► The idea can be adapted to many non-classical calculi e.g. see (Ono 1998). (Kripke, 1959) uses the idea in his decidability proof of $R_{\rightarrow}$.

► Intuition: permute the contraction rules upwards, leaving just the essential ones. Why 3? The conclusion of a left implication rules may require 3 copies.

An upper bound on deciding derivability in BI

▶ Let $2{\uparrow\uparrow}0 = 1$ and let $2{\uparrow\uparrow}(k+1)$ be an exponential tower of 2 of height $k+1$. E.g. $2{\uparrow\uparrow}3 = 2^{2^2}$.

Lemma

Suppose that $F \in \mathsf{Fm}$ has size Θ (i.e. number of symbols). Then

(i) *The number of different sequents that may appear in a derivation of $\mathbf{1} \Rightarrow F$ is $O\big((2{\uparrow\uparrow}(2\Theta+1))^{\Theta}\big)$.*

(ii) *The size of each sequent is $O\big(\Theta^2\big) \cdot O\big((2{\uparrow\uparrow}(2\Theta))^{\Theta+1}\big)$.*

▶ The maximum size of a branch in a backward proof is bounded by taking the product

$$O\big((2{\uparrow\uparrow}(2\Theta+1))^{\Theta}\big) \cdot O\big(\Theta^2\big) \cdot O\big((2{\uparrow\uparrow}(2\Theta))^{\Theta+1}\big)$$

▶ This tower exponential is a space upper bound on deciding derivability (and currently the only complexity bound on BI)