Syntactic Decidability and Complexity Upper Bound for the Logic of Bunched Implication BI

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Abstract. The logic of bunched implication provides a framework for reasoning about resource composition and forms the basis for an assertion language of separation logic. The propositional fragment BI has an elegant proof theory: its bunched calculus combines the sequent calculi for propositional intuitionistic logic and multiplicative intuitionistic linear logic. Hence, BI possesses two implication connectives, one from each logic. Several natural extensions of BI are undecidable, e.g. Boolean BI which replaces intuitionistic logic with classical logic. This makes the decidability of BI, proved via an intricate semantic argument, particularly noteworthy.
A proof-theoretic (syntactic) argument of decidability has thus far proved elusive despite multiple attempts. Here we propose a syntactic proof that is technically interesting, uses a standard bunched calculus, and accessible without any knowledge of the semantics. The decidability argument extends directly to structural rule extensions of BI satisfying easy-to-check syntactic criteria.

1 Introduction

The logic of bunched implication [10,11] provides a logical framework expressive enough to reason about resource composition and systems modelling and forms the basis for an assertion language of separation logic [7] used to reason about software programs. The propositional fragment BI combines the reasoning of intuitionistic logic Ip and multiplicative linear logic MILL. In particular, a BI formula is freely constructed from the intuitionistic (additive) connectives $\vee, \wedge, \rightarrow, \top, \bot$ and the multiplicative connectives $*, -\!\!*, \mathbf{1}$. The decidability of BI [5] was proved via an intricate semantic argument. Kaminski and Francez [8] proposed a syntactic proof, as did Galatos and Jipsen [4], but we have identified crucial gaps in both. Here we propose a syntactic proof of decidability and present the first complexity bounds for the logic.

Bunched calculi [3,9,6,1] were developed to study (substructural) relevant logics, where (as in BI) the derivability of distributivity $A \vee (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$ is desired. The *bunched sequent calculus* LBI_0 for BI is built from sequents $X \Rightarrow A$ where the succedent A is a BI-formula and the antecedent X is a *bunch*, built from BI-formulae using *two* structure connectives, semicolon and comma. The rules of the LBI_0 calculus are essentially the union of the rules of the sequent calculi for Ip and MILL (where both calculi use the same sequent turnstile symbol). Recall that the sequent calculus for Ip is built from sequents $X \Rightarrow A$ where X is a *semicolon*-separated list of Ip formulae and A is an Ip formula. The semicolon is interpreted as $\wedge$ and has commutative, associative, weakening and contraction properties, with identity element $\varnothing_a$. The sequent calculus for MILL is built from sequents $X \Rightarrow A$ where X is a *comma*-separated list of MILL formulae and A is an MILL formula. The comma is interpreted as $*$ and it is commutative and associative with identity element $\varnothing_m$. The two implication connectives are a distinctive feature of BI compared to the relevant logics and indeed, from a resource-logic perspective, to Linear Logic. Here are the left and right rules for $\rightarrow$ and $-\!*$.

$$\frac{Y \Rightarrow C \qquad \Gamma[D] \Rightarrow A}{\Gamma[Y; C \rightarrow D] \Rightarrow A}\ (\rightarrow l) \quad \frac{X; A \Rightarrow B}{X \Rightarrow A \rightarrow B}\ (\rightarrow \mathrm{r}) \quad \frac{Y \Rightarrow C \qquad \Gamma[D] \Rightarrow A}{\Gamma[Y, C -\!* D] \Rightarrow A}\ (-\!* l) \quad \frac{X, A \Rightarrow B}{X \Rightarrow A -\!* B}\ (-\!* \mathrm{r})$$

The remaining rules are as one would expect and yield an elegant proof-theory for BI.

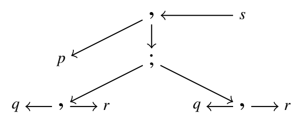
A semantic proof [5] of decidability of the decidability of BI was presented using resource tableaux. The proof there is intricate and requires the development of a large semantic framework, including objects "[to reflect] the information that can be derived from a given set of assumptions" and to built countermodels. Indeed the authors observe: "The relationships identified between resources, labels, dependency graphs, proof-search and resource semantics are central in this study [to prove decidability and finite model property]." Certainly, [5] contains valuable results on BI including decidability and the finite model property. Nevertheless, that proof is less accessible and difficult to check for those not familiar with the semantic methods of BI.

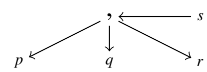
An alternative method of proving decidability is to rely solely on the syntactic characterisation, i.e. the LBI_0 calculus. Aside from its intrinsic technical interest, the attraction of such a syntactic proof is that it enables the argument to be checked without any reference to the semantics. Moreover, the typical combinatorial arguments often yield complexity upper bounds. The usual idea is to show that backward proof search in the calculus from any sequent $\mathbf{1} \Rightarrow F$ can be terminated finitely such that a derivation, if it exists, is guaranteed to be encountered at termination. The specific difficulty for LBI_0 is bounding the maximum nesting of semicolon and commas that need occur in a bunch (to ensure that the backward proof search is complete and has finite termination).

Kaminski and Francez [8] claim that the depth of a bunch (the deepest nesting of commas in its grammar tree) is bounded by the total number $\#_{\mathrm{NL}}(F)$ of Lambek connective occurrences ($*$ and $-\!*$) in the endsequent $\mathbf{1} \Rightarrow F$. Their argument in brief is that every comma along the branch of a bunch is generated by a ($*$l) or ($-\!*$r) rule (viewed from conclusion to premise) and that every comma maps 1-to-1 to a Lambek occurrence in $\mathbf{1} \Rightarrow F$. They do not offer a proof that this map is 1-to-1. Unfortunately, this is the key statement and it requires a careful proof, especially since the contraction rule can 'merge' multiple commas in the premise to a single comma in the conclusion.

Galatos and Jipsen [4] extend the grammar tree of a bunched sequent to define a directed graph. Their crucial claim is that the *multiplicative length*—the maximum (taken over all directed paths in the directed graph) number of negative polarity $*$ and commas and positive polarity $-\!*$—is non-increasing from the conclusion to the premise(s). They treat comma as a n-ary connective rather than a binary connective, else the associative rule (as-c) for comma would already violate their claim since e.g. $((p, q), r), s \Rightarrow t$ would have greater multiplicative length than $(p, q), (r, s) \Rightarrow t$. However, treating comma as a n-ary connective turns out to be problematic as well: for the rule instance below left, the directed graphs for the premise and conclusion are below centre and right, respectively. The multiplicative length for the premise is 2 and for the conclusion it is 1, invalidating the claim in [4].

$$\frac{p,((q,r);(q,r)) \Rightarrow s}{p,q,r \Rightarrow s}\ (\text{ctr})$$





Here we propose a proof of the Kaminski-Francez bound. To achieve this, we first label every comma that occurs in every sequent in the derivation of $\mathbf{1} \Rightarrow F$, so that every comma can be mapped to the Lambek connective in F whence it originated. *If* the deepest nesting of commas in some bunch in the derivation exceeds $\#_{\mathrm{NL}}(F)$, then there must be two labelled commas on the same branch—and hence one comma is nested by the other comma—which originate from the same Lambek connective occurrence $\heartsuit$ in $\mathbf{1} \Rightarrow F$ ($\heartsuit$ has a label in the interval $[1, \#_{\mathrm{NL}}(F)]$). Somewhere above $\mathbf{1} \Rightarrow F$, this single label must have become two labels in the (grammar tree of the) premise of a contraction rule. Since contraction takes X to $X;X$, the least common ancestor (lca) of the two labels must be a semicolon. We track how the lca of the pair of labels can alternate between comma and semicolon and use a parity argument to prove that the above situation cannot occur. Decidability and complexity upper bounds follow.

Recently [2], cutfree bunched calculi for a large class of extensions of BI were obtained by generating new structural rules. Our decision procedure extends directly to those rules where every premise comma has a conclusion comma in the 'same position'.

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