On an equivalence between integral and involutive residuated structures

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Introduction

The aim of the talk is to present some recent work on categorical equivalences for residuated lattices in collaboration with Nick Galatos.

Residuated lattices subsume many natural lattice-ordered algebras: Boolean algebras, Brouwerian algebras, ℓ-groups, MV-algebras, ...

Main theme: can we reconstruct big structures from their smaller parts?

We identify some important cases where this is possible. Moreover, the categories of "big" and "small" objects will turn out to be **equivalent**.

Example: ℓ-groups

Consider the lattice-ordered group of integers $\mathbf{Z} = \langle \mathbb{Z}, \wedge, \vee, +, 0, - \rangle$.

The positive cone of $\mathbf{Z}$ is the monoid of naturals $\mathbf{N} = \langle \mathbb{N}, \wedge, \vee, +, 0, \dot{-} \rangle$.

Here $\dot{-}$ denotes truncated subtraction: $a \dot{-} b = (a - b) \vee 0$.

There are essentially two ways of reconstructing $\mathbf{Z}$ from $\mathbf{N}$:

- the group of fractions construction
- the twist product construction

The group of fractions does not rely on the lattice structure.

The twist product is well suited to lattice-ordered structures.

Example: ℓ-groups

Group of fractions:

- take the set of all pairs $\langle a, b\rangle \in \mathbf{N}^2$, interpreted as $a - b$
- define an algebraic structure on this set:

$$\langle a, b\rangle + \langle c, d\rangle = \langle a + c, b + d\rangle \qquad -\langle a, b\rangle = \langle b, a\rangle$$

- identify those pairs which are equivalent in a suitable sense:

$$\langle a, b\rangle \sim \langle c, d\rangle \iff a + d = b + c$$

Fact: this construction embeds each commutative monoid in a group.

Example: ℓ-groups

Twist product:

- take only normal pairs $\langle a, b \rangle \in \mathbf{N}^2$ defined by a suitable equation:

$$a \wedge b = 0 \text{ (i.e. either } a = 0 \text{ or } b = 0)$$

- figure out how to transform arbitrary pair into equivalent normal pairs:

$$\langle a, b \rangle \mapsto \pi\langle a, b \rangle = \langle a \dot{-} b, b \dot{-} a \rangle$$

- define an algebraic structure on the set of all normal pairs:

$$\langle a, b \rangle + \langle c, d \rangle = \pi\langle a + c, b + d \rangle \quad \langle a, b \rangle \wedge \langle c, d \rangle = \pi\langle a \wedge c, b \vee d \rangle$$

$$-\langle a, b \rangle = \langle b, a \rangle \quad \langle a, b \rangle \vee \langle c, d \rangle = \pi\langle a \vee c, b \wedge d \rangle$$

Example: Sugihara monoids

We can also consider a radically different structure on the integers.

Consider the odd Sugihara monoid $\mathbf{S} = \langle \mathbb{Z}, \wedge, \vee, \cdot, 0, +, 0, - \rangle$ with:

$$x \cdot y = \begin{cases} y \text{ if } |x| < |y| \\ x \text{ if } |x| > |y| \\ x \wedge y \text{ if } |x| = |y| \end{cases} \qquad x + y = \begin{cases} y \text{ if } |x| < |y| \\ x \text{ if } |x| > |y| \\ x \vee y \text{ if } |x| = |y| \end{cases}$$

Observe that the restrictions of these to the negative cone are:

$$x \cdot y = x \wedge y = x + y$$

Example: Sugihara monoids

The negative cone of **S** is the algebra $\mathbf{G} = \langle -\mathbb{N}, \wedge, \vee, \rightarrow \rangle$ with

$$x \rightarrow y = \begin{cases} 1 \text{ if } x \leq y \\ y \text{ if } y < x \end{cases}$$

The odd Sugihara monoid **S** can be reconstructed as a twist product of **G** with the defining equation $a \vee b = 0$ and the normalizing term:

$$\pi\langle a, b\rangle = \langle (a \rightarrow b) \rightarrow a, a \rightarrow b \rangle$$

This construction generalizes to arbitrary Sugihara monoids. In that case we add to the negative cone a constant 0 such that $[0, 1]$ is a Boolean lattice.

Equivalences between integral and involutive structures

Theorem: there are categorical equivalences between the varieties below via a negative functor and a twist product functor:

- Abelian ℓ-groups and integral cancellative divisible residuated lattices ('03 Bahls, Cole, Galatos, Jipsen, Tsinakis)
- odd Sugihara monoids and relative Stone alg's (Gödel alg's without bottom) ('12 Galatos & Raftery)
- Sugihara monoids and relative Stone alg's with a Boolean constant (Fussner & Galatos)

Equivalences between integral and involutive structures

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Theorem (Galatos & Přenosil): the above are instances of a single categorical equivalence between certain involutive and integral structures.

Brouwerian algebras as negative cones of InRLs

Theorem (special case): there is a categorical equivalence between Brouwerian algebras with a Boolean constant and Sugihara-like monoids.

Corollary: between Brouwerian algebras and odd Sugihara-like monoids.

Brouwerian algebras are Heyting algebras without the bottom constant.

Sugihara monoids are distributive idempotent involutive RLs.

Sugihara-like monoids are idempotent involutive RLs with $x(1 \wedge x) = x$.

This is equivalent to the equality $x = (1 \wedge x)(0 \vee x)$.

Corollary: Brouwerian algebras with (without) a Boolean constant are precisely the negative cones of (odd) idempotent involutive RLs.

A little bit about the proof...

Involutive RLs can be thought of as having two monoidal operations $x \cdot y$ and $x + y$ with units 1 and 0 connected by the hemidistributive law:

$$a \cdot (b + c) \leq (a \cdot b) + c$$

as well as a complement operation $\overline{x}$ such that:

$$x \cdot \overline{x} \leq 0 \qquad \text{and} \qquad 1 \leq x + \overline{x}$$

Hemidistributivity ensures that the uniqueness of complements.

That is, involutive RLs are complemented ℓ-bimonoids.

A little bit about the proof. . .

The proof of the main result now proceeds in two steps:

- embed an arbitrary ℓ-bimonoid into a complemented one
- identify cases where this reduces to a twist product

There is a canonical way of doing the first step, generalizing both the group of fractions of a monoid and the Boolean extension of a distributive lattice.

Each element of the complemented extension will have the forms:

$$x = \bigvee_{i \in I} a_i \overline{b_i} \qquad \text{as well as} \qquad x = \bigwedge_{i \in I} a_i + \overline{b_i}$$

In some special cases, these will reduce to $x = a\overline{b}$ and $x = a + \overline{b}$.

A little bit about the proof…

Take an algebra which is precisely the negative cone of its complemented extension. Suppose moreover that the complemented extension satisfies:

$$x = (1 \wedge x) \cdot (0 \vee x), \text{ i.e. } x = (1 \wedge x) \cdot \overline{(1 \wedge \overline{x})}$$

We can then represent its elements as pairs of elements of the neg. cone:

$$\langle 1 \wedge x, 1 \wedge \overline{x} \rangle$$

Such pairs of elements (of the original algebra) will be called normal.

A little bit about the proof...

We are well on our way to defining a twist product construction.

All that remains now is to:

- find an intrinsic description of normal pairs:

$$a \to b = b \qquad b \to a = a \qquad a \cdot b \leq 0$$

- given normal rep's of x and y, compute rep's of $x \cdot y$, $x + y$, etc.:

$$\langle a, b \rangle \cdot \langle c, d \rangle \sim \langle ac, b + d \rangle$$

- transform an arbitrary representation $x = a\overline{b}$ into a normal one:

$$\pi \langle a, b \rangle = \langle (x \to xy0) \to x, x \to y0 \rangle$$

Conclusion

Residuated lattices subsume many natural lattice-ordered algebras: Brouwerian algebras, ℓ-groups, Sugihara monoids, ...

We can often make a trade-off between integrality and involutivity.

In many important cases an involutive RL can be reconstructed from its negative cone by means of a suitable twist product.

To prove that the twist product works as intended, it is useful to:

- construct the complemented extension ($x = \bigvee_i a_i\overline{b_i}$)
- then show that it simplifies ($x = a\overline{b}$)

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Thank you for your attention.