Hybrid and Subexponential Linear Logics

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Motivation: LL as a logical Framework

- Resource conscious.
- Strong foundations and meta-theory.
- Practical and theoretical machinery: tools, conditions for cut-elimination of object logics, etc.

Applications

- Encodings of other logics.
- Specification and verification of (concurrent) systems.

Drawback

Some specifications require modalities different from those of LL, e.g., :

- Time : actions happen in a given time-unit.
- Spatial information : agents are confined to locations.

Capturing other modalities in LL

Hybrid and subexponential LL

Temporal dimension : transitions may have durations:

$$state(t) \longrightarrow_d state'(t+d)$$

HyLL (Hybrid Linear Logic)

Truth judgments are labeled by worlds

- $\Gamma; \Delta \vdash F@w$ meaning F holds at world w.
- Worlds have a monoidal structure (e.g., $Time = (\mathcal{N}, +, 0)$)
- Hybrid connectives (e.g., at) relate worlds with formulas.

$$(\forall t.state \texttt{ at } t \multimap state' \texttt{ at } d.t)@1; state@5 \vdash state'@d.5$$

Capturing other modalities in LL

Hybrid and subexponential LL

Spatial behavior : agents confined to spaces/locations:

$$[A]_a \mid [B]_b \mid [0]_c$$

SELL: Subexponentials in linear logic

- $!, ?$ are not canonical. In SELL, they are decorated with indexes.
- Indexes are organized in a pre-order (determining the provability relation).
- According to the pre-order, $!^a$ and $?^a$ may have different interpretations: time-units, locations, spaces, preferences, etc .

In this talk

We study the relative expressiveness power of HyLL and SELL

- We show that HyLL can deeply encoded into LL (and hence into SELL).
- We give another encoding from HyLL into SELL that gives better insights on the meaning of words in HyLL.

… as a side dish

We show two characterizations of CTL into LL + fixed points:

- Into μMALL, given an "operational" interpretation of CTL: states are linear atoms controlled by formulas representing transitions.
- Into μHyMALL where states are words, thus following closer the semantics of CTL.

Outline

1. HyLL
2. SELL
3. CTL in Linear Logic + Fixed Points

Intuitionistic Linear Logic

Dyadic System

- Sequents are of the form: $\Gamma; \Delta \vdash C$, where
  Γ is a set of formulas , called the *unrestricted / classical context*
  Δ is a multiset of formulas, called the *linear context*
- Init and Copy

$$\Gamma; p \vdash p \ \mathit{init} \qquad \frac{\Gamma, A; \Delta, A \vdash C}{\Gamma, A; \Delta \vdash C} \ \mathit{copy}$$

- Exponentials:

$$\frac{\Gamma; . \vdash A}{\Gamma; . \vdash !A} \ !R \qquad \frac{\Gamma, A; \Delta \vdash C}{\Gamma; \Delta, !A \vdash C} \ !L$$

Dyadic System for ILL (2)

- Multiplicatives:

$$\frac{\Gamma; \Delta, A \vdash B}{\Gamma; \Delta \vdash A \multimap B} [\multimap_R] \qquad \frac{\Gamma; \Delta \vdash A \quad \Gamma; \Delta', B \vdash C}{\Gamma; \Delta, \Delta', A \multimap B \vdash C} [\multimap_L]$$

$$\frac{\Gamma; \Delta \vdash A \quad \Gamma; \Delta' \vdash B}{\Gamma; \Delta, \Delta' \vdash A \otimes B} \otimes R \quad \frac{\Gamma; \Delta, A, B \vdash C}{\Gamma; \Delta, A \otimes B \vdash C} \otimes L$$

- Additives:

$$\frac{\Gamma; \Delta \vdash A \qquad \Gamma; \Delta \vdash B}{\Gamma; \Delta \vdash A \,\&\, B} \,\&\, R \qquad \frac{\Gamma; \Delta, A_i \vdash C}{\Gamma; \Delta, A_1 \,\&\, A_2 \vdash C} \,\&\, L_i$$

$$\frac{\Gamma; \Delta \vdash A_i}{\Gamma; \Delta \vdash A_1 \oplus A_2} \oplus R_i \qquad \frac{\Gamma; \Delta, A \vdash C \quad \Gamma; \Delta, B \vdash C}{\Gamma; \Delta, A \oplus B \vdash C} \oplus L$$

Hybrid Linear Logic

The idea: to add constraint reasoning to LL (transition systems that operate under temporal, stochastic or probabilistic constraints).

- Conservative extension of (I)LL.
- $A@w$ means that A holds at world w.

Definition

A *constraint domain* $\mathcal{W}$ is a monoid structure $\langle W, ., \iota\rangle$. The elements of W are called worlds.
The partial order $\preceq : W \times W$ ($u \preceq w$ if there exists $v \in W$ s.t. $u.v = w$) is the reachability relation .

- An example: Time: $\mathcal{T} = \langle \boldsymbol{N}, +, 0\rangle$.
- The identity world ι, $\preceq$-initial, represents the lack of any constraints.
- ILL $\subseteq$ HyLL[ι] $\subset$ HyLL[W].

Hybrid Linear Logic

- Sequents are of the form: $\Gamma; \Delta \vdash C \;@\; w$.
- Γ and Δ are multisets of judgments of the form $A \;@\; w$
- All ordinary rules continue essentially unchanged.

$$\Gamma; p \;@\; w \vdash p \;@\; w \; init \qquad \frac{\Gamma, A \;@\; w; \Delta, A \;@\; w \vdash C \;@\; w}{\Gamma, A \;@\; w; \Delta \vdash C \;@\; w} \; copy$$

- Exponentials rules

$$\frac{\Gamma; . \vdash A \;@\; w}{\Gamma; . \vdash !A \;@\; w} \; !R \qquad \frac{\Gamma, A \;@\; u; \Delta \vdash C \;@\; w}{\Gamma; \Delta, !A \;@\; u \vdash C \;@\; w} \; !L$$

- Multiplicative rules

$$\frac{\Gamma; \Delta \vdash A \;@\; w \quad \Gamma; \Delta' \vdash B \;@\; w}{\Gamma; \Delta, \Delta' \vdash A \otimes B \;@\; w} \otimes R \qquad \frac{\Gamma; \Delta, A \;@\; u, B \;@\; u \vdash C \;@\; w}{\Gamma; \Delta, A \otimes B \;@\; u \vdash C \;@\; w} \otimes L$$

- Additive rules

$$\frac{\Gamma; \Delta \vdash A \;@\; w \qquad \Gamma; \Delta \vdash B \;@\; w}{\Gamma; \Delta \vdash A \,\&\, B \;@\; w} \;\& R$$

Hybrid Connectives

$$A, B, \ldots ::= \ldots \mid A \texttt{ at } w \mid \downarrow u.\ A \mid \forall u.\ A \mid \exists u.\ A$$

- To introduce the satisfaction proposition $(A \texttt{ at } u)$ (at any world v), the proposition A must be true in the world u:

$$\frac{\Gamma; \Delta \vdash A \ @\ u}{\Gamma; \Delta \vdash (A \texttt{ at } u) \ @\ v} \texttt{ at } R \qquad \frac{\Gamma; \Delta, A \ @\ u \vdash C \ @\ w}{\Gamma; \Delta, (A \texttt{ at } u) \ @\ v \vdash C \ @\ w} \texttt{ at } L$$

- The localization $\downarrow u.\ A$ binds u to the current world :

$$\frac{\Gamma; \Delta \vdash [w/u]A \ @\ w}{\Gamma; \Delta \vdash \downarrow u.A \ @\ w} \downarrow R \qquad \frac{\Gamma; \Delta, [v/u]A \ @\ v \vdash C \ @\ w}{\Gamma; \Delta, \downarrow u.A \ @\ v \vdash C \ @\ w} \downarrow L$$

- Quantifiers have the usual rules (α is fresh in $\exists L$):

$$\frac{\Gamma; \Delta \vdash A[\tau/\alpha]@w}{\Gamma; \Delta \vdash \exists\alpha.A@w} \exists R \qquad \frac{\Gamma; \Delta, A@w \vdash C@v}{\Gamma; \Delta, \exists\alpha.A@w \vdash C@v} \exists L$$

Defining Modal Connectives

- Defined modal connectives:

$$\Box A \stackrel{\text{def}}{=} \downarrow u.\ \forall w.\ (A \texttt{ at } u.w) \qquad \Diamond A \stackrel{\text{def}}{=} \downarrow u.\ \exists w.\ (A \texttt{ at } u.w)$$
$$\delta_v\, A \stackrel{\text{def}}{=} \downarrow u.\ (A \texttt{ at } u.v) \qquad \dagger\, A \stackrel{\text{def}}{=} \forall u.\ (A \texttt{ at } u)$$

- The connective δ represents a form of *delay*:
  Derived right rule:

$$\frac{\Gamma; \Delta \vdash A \mathbin{@} w.v}{\Gamma; \Delta \vdash \delta_v\, A \mathbin{@} w}\ \delta\ R$$

Theorem (Admissibility of cut)

If $\Gamma; \Delta \vdash A \mathbin{@} u$ *and* $\Gamma; \Delta', A \mathbin{@} u \vdash C \mathbin{@} w$, *then* $\Gamma; \Delta, \Delta' \vdash C \mathbin{@} w$.

Encodings in Linear Logic

Two meta-level predicates $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ for identifying objects that appear on the left or right side of the sequents in the object logic. Rules

$$\frac{\Delta, A_i \longrightarrow \Gamma}{\Delta, A_1 \wedge A_2 \longrightarrow \Gamma} \wedge_{Li} \quad \frac{\Delta \longrightarrow \Gamma, A \quad \Delta \longrightarrow \Gamma, B}{\Delta \longrightarrow \Gamma, A \wedge B} \wedge_R$$

are specified in LL as

$$\wedge_L : \exists A_1, A_2.(\lfloor A_1 \wedge A_2 \rfloor^\perp \otimes (\lfloor A_1 \rfloor \oplus \lfloor A_2 \rfloor))$$

$$\wedge_R : \exists A, B.(\lceil A \wedge B \rceil^\perp \otimes (\lceil A \rceil \;\&\; \lceil B \rceil))$$

The linear logic connectives indicate how these object level formulas are connected: contexts are copied (&) or split ($\otimes$), in different inference rules ($\oplus$) or in the same sequent ($\parr$).

HyLL and Linear Logic

HyLL rules can be encoded in LL as:

$$
\begin{array}{lcl}
\otimes\ R & : & \exists C, C', w.(\lceil (C \otimes C')@w \rceil^{\perp} \otimes \lceil C@w \rceil \otimes \lceil C'@w \rceil) \\
\otimes\ L & : & \exists C, C', w.(\lfloor (C \otimes C')@w \rfloor^{\perp} \otimes (\lfloor C@w \rfloor \mathbin{⅋} \lfloor C'@w \rfloor)) \\
\texttt{at}\ R & : & \exists C, u, w.(\lceil (C\ \texttt{at}\ u)@w \rceil^{\perp} \otimes \lceil C@u \rceil) \\
\texttt{at}\ L & : & \exists C, u, w.(\lfloor (C\ \texttt{at}\ u)@w \rfloor^{\perp} \otimes \lfloor C@u \rfloor) \\
\downarrow R & : & \exists A, u, w.(\lceil \downarrow u.A@w \rceil^{\perp} \otimes \lceil (A\ w)@w \rceil) \\
\downarrow L & : & \exists A, u, w.(\lfloor \downarrow u.A@w \rfloor^{\perp} \otimes \lfloor (A\ w)@w \rfloor)
\end{array}
$$

Theorem (Adequacy)

$\Gamma; \Delta \vdash F@w$ *iff* $\vdash ?\Upsilon, ?\lfloor \Gamma \rfloor, \lfloor \Delta \rfloor, \lceil F@w \rceil$.

Adequacy on the level of derivations : focusing on a LL specification clause, the (bipole) derivation corresponds exactly to applying the introduction rule of HyLL.

Subexponentials in Linear Logic

Subexponential Signature

$\Sigma = \langle I, \preceq, U \rangle$ where I is a set of labels, $U \subseteq I$ set of unbounded subexp and $\preceq$ is a pre-order among the elements of I.

$$\frac{\Gamma, F \longrightarrow G}{\Gamma, !^a F \longrightarrow G}\ !^a{}_L \qquad \frac{!^{a_1} F_1, \ldots, !^{a_n} F_n \longrightarrow F}{!^{a_1} F_1, \ldots, !^{a_n} F_n \longrightarrow !^a F}\ !^a{}_R, \text{ provided } a \preceq a_i$$

$$\frac{\Gamma \longrightarrow G}{\Gamma, !^b F \longrightarrow G}\ W \qquad \frac{\Gamma, !^b F, !^b F \longrightarrow G}{\Gamma, !^b F \longrightarrow G}\ C \qquad \text{provided } b \in U$$

Classical Dyadic System

$$\frac{\vdash a_1 : \Gamma_1, \ldots, a_i : \Gamma_i \star \{F\}, \ldots, a_n : \Gamma_n ; \Delta}{\vdash a_1 : \Gamma_1, \ldots, a_n : \Gamma_n ; \Delta, ?^{a_i} F}\ ?$$

$\star$ can be $\cup$ or $\uplus$ depending whether $a_i \in U$ or not.

Quantification on Subexponentials

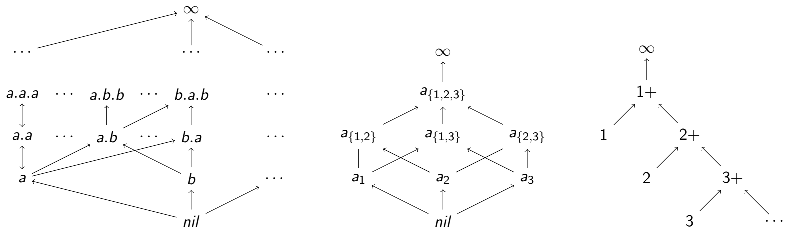
$$\frac{\mathcal{A}; \mathcal{L}; \Gamma, P[l/x] \vdash G}{\mathcal{A}; \mathcal{L}; \Gamma, \Cap x : a.P \vdash G} \Cap_L \quad \frac{\mathcal{A}, l_e : a; \mathcal{L}; \Gamma \vdash P[l_e/x]}{\mathcal{A}; \mathcal{L}; \Gamma \vdash \Cap x : a.P} \Cap_R$$

- Creating "new" locations: $\Gamma, \Cup l.(F) \vdash G$
- Asserting something about all locations: $\Gamma, \Cap l.(F) \vdash G$
- Proving that all locations satisfies G: $\Gamma \vdash \Cap l.(G)$
- Proving that G holds in some location: $\Gamma \vdash \Cup l.(G)$

Cut-elimination and Focusing

For any signature Σ, the proof system $\mathrm{SELL}^{\Cap}$ admits cut-elimination and a complete focused proof system.

Concurrent Behaviors in $\mathrm{SELL}^{\Cap}$



Connective	Meaning
$\bigtriangledown_s = !^s$	P is located at s.
$\bigtriangledown_s = !^s ?^s$	P is confined to s.
$\Cap l : a\ P$	P can move to locations "below" a

- Epistemic, spatial and temporal modalities can be specified in $\mathrm{SELL}^{\Cap}$.
- Considering C-semiring signatures, also preferences (fuzzy, probabilities, costs, etc) can be specified.
- Bigraphs have been also encoded into SELL.

HyLL and SELL

A HyLL formula $F@w$ is stored as $?^w F$ (in the w context of SELL)

$$
\begin{array}{lcl}
\otimes\ R & : & \exists C, C'.\, \Cup\, w : \infty.(!^w\lceil (C \otimes C')@w \rceil^\perp \otimes ?^w\lceil C@w \rceil \otimes ?^w\lceil C'@w \rceil) \\
\mathtt{at}\ R & : & \exists A.\, \Cup\, u : \infty, w : \infty.(!^w\lceil (A\ \mathtt{at}\ u)@w \rceil^\perp \otimes ?^u\lceil A@u \rceil) \\
\mathtt{at}\ L & : & \exists A.\, \Cup\, u : \infty, w : \infty.(!^w\lfloor (A\ \mathtt{at}\ u)@w \rfloor^\perp \otimes ?^u\lfloor A@u \rfloor) \\
\downarrow R & : & \exists A.\, \Cup\, u : \infty, w : \infty.(!^w\lceil \downarrow u.A@w \rceil^\perp \otimes ?^w\lceil (A\ w)@w \rceil) \\
\downarrow L & : & \exists A.\, \Cup\, u : \infty, w : \infty.(!^w\lfloor \downarrow u.A@w \rfloor^\perp \otimes ?^w\lfloor (A\ w)@w \rfloor)
\end{array}
$$

Theorem (Adequacy)

$\Gamma; \Delta \vdash F@w$ *iff* $\vdash ?^c\Upsilon, ?^c\lfloor \Gamma \rfloor, \lfloor \Delta \rfloor, ?^w\lceil F@w \rceil$.

- *Adequacy at the level of derivations.*
- *For any domain $\mathcal{W}$, we only need a flat SELL signature (plus ∞ and c where for all $w \in \mathcal{W}$, $w \preceq \infty$).*

Information Confinement

In SELL

- inconsistency is local: $!^w?^w\mathbf{0} \not\vdash \mathbf{0}$
- inconsistency is not propagated: $!^w?^w\mathbf{0} \not\vdash !^v?^v\mathbf{0}$

In HyLL this is not possible. The rule for 0 is

$$\frac{}{\Gamma; \Delta, \mathbf{0}@w \vdash F@v}\ \mathbf{0}L$$

Even if we consider a "weaker" rule:

$$\frac{}{\Gamma; \Delta, \mathbf{0}@w \vdash F@w}\ \mathbf{0}L - \mathit{weak}$$

$\mathbf{0}L$ is still admissible:

$$\frac{\Gamma; \Delta, \mathbf{0}@w \vdash (\mathbf{0}\ \mathtt{at}\ v)@w \quad 0_L \qquad \dfrac{\Gamma; \Delta, \mathbf{0}@v \vdash F@v \quad 0_L}{\Gamma; \Delta, (\mathbf{0}\ \mathtt{at}\ v)@w \vdash F@v}\ at_L}{\Gamma; \Delta, \mathbf{0}@w \vdash F@v}\ \mathrm{cut}$$

Computational Tree Logic

Syntax

Path quantifiers $\mathbf{Q} \in \{\mathsf{A}, \mathsf{E}\}$ and formulas:

$$F \quad ::= \quad p \mid F \wedge F \mid F \vee F \mid \mathbf{Q}\mathsf{X}F \mid \mathbf{Q}\mathsf{F}F \mid \mathbf{Q}\mathsf{G}F \mid \mathbf{Q}[F \mathsf{U} F]$$

Fixpoint characterization of CTL connectives

$$\begin{aligned} \mathsf{EF}F &= \mu Y.F \vee \mathsf{EX}Y & \qquad \mathsf{EG}F &= \nu Y.F \wedge \mathsf{EX}Y \\ \mathsf{AF}F &= \mu Y.F \vee \mathsf{AX}Y & \qquad \mathsf{AG}F &= \nu Y.F \wedge \mathsf{AX}Y \end{aligned}$$

Two encodings:

- Into μMALL (more "operational" but not quite compositional).
- Into μHyMALL (pleasant duality with CTL semantics).

Fixed points in LL

μMALL system

$$\frac{\Sigma \vdash \Delta, S\vec{t} \qquad \vec{x} \vdash B\ S\vec{x}, (S\vec{x})^{\perp}}{\Sigma \vdash \Delta, \nu B\vec{t}}\ \nu \qquad \frac{\Sigma \vdash \Delta, B(\mu B)\vec{t}}{\Sigma \vdash \Delta, \mu B\vec{t}}\ \mu$$

- S is the (co)inductive invariant.
- μ corresponds to unfolding
- ν allows for (co)induction.

The system μHyMALL (*Least fixed point rules*)

$$\frac{\Sigma; \Delta \vdash B(\mu B)\vec{t} \ @\ w}{\Sigma; \Delta \vdash \mu B\vec{t} \ @\ w}\ \mu\ R$$

$$\frac{\vec{x}; \cdot; BS\vec{x} \ @\ u \vdash S\vec{x} \ @\ u \quad \Sigma; \Delta, S\vec{t} \ @\ u \vdash C \ @\ w}{\Sigma; \Delta, \mu B\vec{t} \ @\ u \vdash C \ @\ w}\ \mu\ L$$

Encoding into μMALL

R is a finite set of state transitions.

$$\mathcal{C}[\![\mathrm{AF}F]\!]_{\mathcal{R}} = \mu Y.\, \mathcal{C}[\![F]\!]_{\mathcal{R}} \oplus \mathop{\&}_{(\mathtt{s},\mathtt{s}')\in R} (\mathtt{neg}(\mathtt{s}) \oplus (\mathtt{pos}(\mathtt{s}) \otimes (\mathcal{C}[\![\mathtt{s}']\!]_{\mathcal{R}} \parr Y))$$

$$\mathcal{C}[\![\mathrm{EF}F]\!]_{\mathcal{R}} = \mu Y.\, \mathcal{C}[\![F]\!]_{\mathcal{R}} \oplus \bigoplus_{(\mathtt{s},\mathtt{s}')\in R} (\mathtt{pos}(\mathtt{s}) \otimes (\mathcal{C}[\![\mathtt{s}']\!]_{\mathcal{R}} \parr Y))$$

$$\mathcal{C}[\![\mathrm{AG}F]\!]_{\mathcal{R}} = \nu Y.\, \mathcal{C}[\![F]\!]_{\mathcal{R}} \,\&\, \mathop{\&}_{(\mathtt{s},\mathtt{s}')\in R} (\mathtt{neg}(\mathtt{s}) \oplus (\mathtt{pos}(\mathtt{s}) \otimes (\mathcal{C}[\![\mathtt{s}']\!]_{\mathcal{R}} \parr Y))$$

$$\mathcal{C}[\![\mathrm{EG}F]\!]_{\mathcal{R}} = \nu Y.\, \mathcal{C}[\![F]\!]_{\mathcal{R}} \,\&\, \bigoplus_{(\mathtt{s},\mathtt{s}')\in R} (\mathtt{pos}(\mathtt{s}) \otimes (\mathcal{C}[\![\mathtt{s}']\!]_{\mathcal{R}} \parr Y))$$

Theorem (Adequacy)

Let $\mathcal{K} = \langle S, I, R, L\rangle$ be a Kripke structure on a set of atomic propositions $\mathcal{P}$, $s \in S$ be a state and F be a CTL formula. Then

$$s \models^{\mathcal{K}}_{CTL} F \;\; \textit{iff} \;\; \vdash \mathcal{C}[\![s]\!]_{\mathcal{R}}, \mathcal{C}[\![F]\!]_{\mathcal{R}}$$

Encoding into μHyMALL

The transition system: A non-recursive lfp expression

$$\texttt{trans} \triangleq \mu\Big(\lambda T.\, \lambda u.\, \lambda v.\, \bigoplus_{(s,s')\in R} (s = u \otimes s' = v)\Big).$$

$$\begin{aligned}
[\![\mathrm{AX}F]\!] &= {\downarrow} u.\, \forall w.\, \texttt{trans}\ u\ w \otimes ([\![F]\!]\ \texttt{at}\ w) \\
[\![\mathrm{EX}F]\!] &= {\downarrow} u.\, \exists w.\, \texttt{trans}\ u\ w \otimes ([\![F]\!]\ \texttt{at}\ w) \\
[\![\mathrm{AG}F]\!] &= \nu(\lambda Y.\, [\![F]\!] \,\&\, {\downarrow} u.\, \forall w.\, \texttt{trans}\ u\ w \otimes (Y\ \texttt{at}\ w)) \\
[\![\mathrm{EG}F]\!] &= \nu(\lambda Y.\, [\![F]\!] \,\&\, {\downarrow} u.\, \exists w.\, \texttt{trans}\ u\ w \otimes (Y\ \texttt{at}\ w))
\end{aligned}$$

Theorem (Adequacy)

Let $\mathcal{K} = \langle S, I, R, L\rangle$ be a Kripke structure on a set of atomic propositions $\mathcal{P}$, $s \in S$ be a state and F be a CTL formula. Then,

$$\cdot;\cdot \vdash [\![F]\!] \ \texttt{@}\ s \quad \textit{iff} \quad s \models^{\mathcal{K}}_{CTL} F$$

For any F, s, $\cdot; [\![\mathrm{AG}F]\!] \ \texttt{@}\ s \vdash [\![\mathrm{AF}F]\!] \ \texttt{@}\ s$

Summing up

- HyLL worlds cannot "control" the logical context as the subexponentials do (promotion rule).
- Now we know that worlds in HyLL do not add any expressiveness power.
- However, HyLL has shown to be more flexible/versatile for some modeling task.
- In fact, HyLL worlds give us a more natural encoding of CTL into LL-frameworks.

Thanks!