Measurable Kleene Algebras and Structural Control

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Overview

Motivations

- (Kurtonina& Moorgat 97) the fact that grammar rules often admit exceptions, understood as rules which yield grammatically non well-formed constructions if applied unrestrictedly, but grammatically well-formed sentences if applied in a controlled way;
- aims at establishing systematic forms of communication between different grammatical regimes in formal linguistics;
- the basic Lambek calculus incarnates the most general grammatical regime, and the special behaviour of its extensions is captured by additional analytic structural rules;
- a systematic two-way communication between these grammatical regimes is captured by introducing extra pairs of adjoint modal operators (the structural control operators).

Examples

- embedding from **L** to **NL** $+\{\Box, \blacklozenge\}$:

$$\begin{array}{lcl} p^{\#} & = & p \\ A \cdot_{\mathbf{L}} B & = & \blacklozenge(A^{\#} \cdot_{\mathbf{NL}} B^{\#}) \\ A/_{\mathbf{L}}B & = & \Box A^{\#}/_{\mathbf{NL}}B^{\#} \\ A\backslash_{\mathbf{L}}B & = & \Box A^{\#}\backslash_{\mathbf{NL}}B^{\#} \end{array}$$

- controlled associativity in **NL** $+\{\Box, \blacklozenge\}$:

$$\blacklozenge(A \cdot_{\mathbf{NL}} \blacklozenge(B \cdot_{\mathbf{NL}} C)) = \blacklozenge(\blacklozenge(A \cdot_{\mathbf{NL}} B) \cdot_{\mathbf{NL}} C)$$

Kleene algebras and continous Kleene algebras

Definition (Kozen90)

A *Kleene algebra* is a structure $\mathbb{K} = (K, \cup, \cdot, ()^*, 1, 0)$ such that:

- K1 $(K, \cup, 0)$ is a join-semilattice with bottom element 0;
- K2 $(K, \cdot, 1)$ is a monoid with unit 1, moreover $\cdot$ preserves $\cup$ in each coordinate, and 0 is an annihilator for $\cdot$;
- K3 $1 \cup \alpha \cdot \alpha^* \leq \alpha^*$, $1 \cup \alpha^* \cdot \alpha \leq \alpha^*$, and $1 \cup \alpha^* \cdot \alpha^* \leq \alpha^*$;
- K4 $\alpha \cdot \beta \leq \beta$ implies $\alpha^* \cdot \beta \leq \beta$;
- K5 $\beta \cdot \alpha \leq \beta$ implies $\beta \cdot \alpha^* \leq \beta$.

A Kleene algebra is *continuous* if:

- K1' $(K, \cup, 0)$ is a *complete* join-semilattice;
- K2' $\cdot$ is *completely* join-preserving in each coordinate;
- K6 $\alpha^* = \bigcup \alpha^n$ for $n \geq 0$.

We focus on the continuous Kleene algebras in the following.

Observation

Since $()^*$ is a closure operator, for any Kleene algebra $\mathbb{K}$, the operation $()^* : K \to K$ is a closure operator on K seen as a poset. By general order-theoretic facts this means that

$$()^* = e\gamma,$$

where $\gamma : K \twoheadrightarrow Range(*)$, defined by $\gamma(\alpha) = \alpha^*$ for every $a \in K$, is the left adjoint of the natural embedding $e : Range(*) \hookrightarrow K$, i.e. for every $\alpha \in K$, and $\xi \in Range(*)$,

$$\gamma(\alpha) \leq \xi \quad \text{iff} \quad \alpha \leq e(\xi).$$

Notice that if we consider γ and e as $\blacklozenge$ and $\Box$, hence for any $\alpha \in K$, $\alpha^* = \Box\blacklozenge\alpha$.

Moving to proof theory

- controlled structure rules to cover the following inequalities:

$$\Box\blacklozenge\alpha \leq \Box\blacklozenge\alpha \cdot \Box\blacklozenge\alpha \text{ and } \Box\blacklozenge\alpha \cdot \Box\blacklozenge\alpha \leq \Box\blacklozenge\alpha$$

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- problematic way:

$$\text{cc}\ \frac{\Box\blacklozenge\alpha \cdot \Box\blacklozenge\alpha \vdash \beta}{\Box\blacklozenge\alpha \vdash \beta} \qquad \text{cex}\ \frac{\Box\blacklozenge\alpha \vdash \beta}{\Box\blacklozenge\alpha \cdot \Box\blacklozenge\alpha \vdash \beta}$$

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- multi-type way: let $S = \{\xi \mid \xi = \blacklozenge\alpha \text{ for some } \alpha \in K\}$,

$$\text{cc}\ \frac{\Box\blacklozenge\alpha \cdot \Box\blacklozenge\alpha \vdash \beta}{\Box\blacklozenge\alpha \vdash \beta} \qquad \text{abs}\ \frac{\alpha_1 \vdash \Box\xi \qquad \alpha_2 \vdash \Box\xi}{\alpha_1 \cdot \alpha_2 \vdash \Box\xi}$$

Measurable Kleene algebras

Definition

A *measurable Kleene algebra* is a structure $\mathbb{K} = (K, \cup, \cdot, ()^*, ()^\star, 1, 0)$ such that:

MK1 $(K, \cup, \cdot, ()^*, 1, 0)$ is a continuous Kleene algebra;

MK2 $()^\star$ is a monotone unary operation;

MK3 $1 \leq \alpha^\star$, and $\alpha^\star \cdot \alpha^\star \leq \alpha^\star$;

MK4 $\alpha^\star \leq \alpha$ and $\alpha^\star \leq \alpha^{\star\star}$;

MK5 $\beta \leq \alpha$ and $1 \leq \beta$ and $\beta \cdot \beta \leq \beta$ implies $\beta \leq \alpha^\star$.

The name is chosen by analogy with measurable sets in analysis, which are defined in terms of the existence of approximations “from above” and “from below”.

Observations

By general order-theoretic facts,

$$()^\star = e'\imath,$$

where $\imath : K \twoheadrightarrow Range(\star)$, defined by $\imath(\alpha) = \alpha^\star$ for every $a \in K$, is the right adjoint of the natural embedding $e' : Range(\star) \hookrightarrow K$, i.e. for every $\alpha \in K$ and $\xi \in Range(\star)$,

$$e'(\xi) \leq \alpha \quad \text{iff} \quad \xi \leq \imath(\alpha).$$

Observations

Lemma

For any measurable Kleene algebra $\mathbb{K}$ and any $\alpha \in K$, if $1 \leq \alpha$ and $\alpha \cdot \alpha \leq \alpha$, then

$$\alpha^* = \alpha = \alpha^\star.$$

Hence,

$$Range(*) = Range(\star) = \{\beta \in K \mid 1 \leq \beta \text{ and } \beta \cdot \beta \leq \beta\}.$$

This guarantees that

$$Range(*) = Range(\star) = \{\beta \in K \mid 1 \leq \beta \text{ and } \beta \cdot \beta \leq \beta\}.$$

Hence, e' coincides with the natural embedding $e : Range(*) \hookrightarrow K$, which is then endowed with both the left adjoint and the right adjoint. We can consider $\imath$ as $\blacksquare$, hence for any $\alpha \in K$, $\alpha^\star = \Box\blacksquare\alpha$.

Proof theory again

- controlled structure rules to cover the following inequalities:

$$\Box\blacklozenge\alpha \leq \Box\blacklozenge\alpha \cdot \Box\blacklozenge\alpha$$

- multi-type way: let $S = \{\xi \mid \xi = \blacklozenge\alpha$ for some $\alpha \in K\} = \{\xi \mid \xi = \blacksquare\alpha$ for some $\alpha \in K\}$,

$$\text{cc}\ \frac{\Box\xi \cdot \Box\xi \vdash \beta}{\Box\xi \vdash \beta}$$

Heterogeneous measurable Kleene algebras

Definition

A *heterogeneous measurable Kleene algebra* is a tuple $\mathbb{H} = (\mathbb{A}, \mathbb{B}, \iota, \gamma, e)$ verifying the following conditions:

- H1 $\mathbb{A} = (A, \sqcup, \cdot, 1_s, 0)$ is such that $(A, \sqcup, 0)$ a complete join-semilattice with bottom element 0 and $(A, \cdot, 1_s)$ a monoid with unit 1, moreover $\cdot$ preserves arbitrary joins in each coordinate, and 0 is an annihilator for $\cdot$;
- H2 $\mathbb{B} = (S, \sqcup, 0_s)$ is a complete join-semilattice with bottom element 0_s;
- H3 $e(\gamma(\alpha)) = \bigcup \alpha^n$ for any $n \in \mathbb{N}$.
- H4 $\gamma : \mathbb{A} \twoheadrightarrow \mathbb{B}$ and $\iota : \mathbb{A} \twoheadrightarrow \mathbb{B}$ and $e : \mathbb{B} \hookrightarrow \mathbb{A}$ are such that $\gamma \dashv e \dashv \iota$ and $\gamma(e(\xi)) = \xi = \iota(e(\xi))$ for all $\xi \in S$;
- H5 $1 \leq e(\xi)$, and $e(\xi) \cdot e(\xi) \leq e(\xi)$ for any $\xi \in \mathbb{B}$;
- H6 For any $\beta \in \mathbb{A}$, if $1 \leq \beta$ and $\beta \cdot \beta \leq \beta$, then $\gamma(\beta) \leq \iota(\beta)$.

From single type to multi-type

Definition

For any measurable Kleene algebra $\mathbb{K} = (K, \cup, \cdot, ()^{*}, ()^{\star}, 1, 0)$, let

$$\mathbb{K}^{+} = (\mathbb{A}, \mathbb{B}, \iota, \gamma, e)$$

be the structure defined as follows:

1. $\mathbb{A} := (K, \cup, \cdot, 1, 0)$ is the $\{()^{*}, ()^{\star}\}$-free reduct of $\mathbb{K}$;
2. $\mathbb{B} := (S, \cup, 0^{*})$;
3. $\gamma : \mathbb{A} \twoheadrightarrow \mathbb{B}$ and $e : \mathbb{B} \hookrightarrow \mathbb{A}$ are defined as the maps into which the closure operator $()^{*}$ decomposes, and $\iota : \mathbb{A} \twoheadrightarrow \mathbb{B}$ and $e : \mathbb{B} \hookrightarrow \mathbb{A}$ are defined as the maps into which the interior operator $()^{\star}$ decomposes.

Proposition

For any measurable Kleene algebra $\mathbb{K}$, the structure $\mathbb{K}^{+}$ defined above is a heterogeneous measurable Kleene algebra.

From multi-type to single type

Definition

For any heterogeneous measurable Kleene algebra $\mathbb{H} = (\mathbb{A}, \mathbb{B}, \iota, \gamma, e)$, let $\mathbb{H}_+ := (\mathbb{A}, ()^*, ()^\star)$, where $()^* : \mathbb{A} \to \mathbb{A}$ and $()^\star : \mathbb{A} \to \mathbb{A}$ are respectively defined by $\alpha^* := e(\gamma(\alpha))$ and $\alpha^\star := e(\iota(\alpha))$ for every $\alpha \in \mathbb{A}$.

Proposition

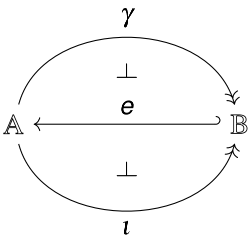
For any heterogeneous measurable Kleene algebra $\mathbb{H} = (\mathbb{A}, \mathbb{B}, \iota, \gamma, e)$, the structure $\mathbb{H}_+$ defined above is a measurable Kleene algebra. Moreover, the kernel of $\mathbb{H}_+$ is join-semilattice-isomorphic to $\mathbb{B}$.

The heterogenous representation of MKAs

Proposition

For any measurable Kleene algebra $\mathbb{K}$ *and heterogeneous measurable Kleene algebra* $\mathbb{H}$*,*

$$\mathbb{K} \cong (\mathbb{K}^{+})_{+} \quad \textit{and} \quad \mathbb{H} \cong (\mathbb{H}_{+})^{+}.$$



- Operational and structural connectives:

General
$$\begin{cases} \alpha ::= a \mid 1 \mid 0 \mid \Box\xi \mid \alpha \cup \alpha \mid \alpha \cdot \alpha \\ \Gamma ::= \Phi \mid \circ\Pi \mid \Gamma \odot \Gamma \mid \Gamma < \Gamma \mid \Gamma > \Gamma \end{cases}$$

Special
$$\begin{cases} \xi ::= \blacklozenge\alpha \mid \blacksquare\alpha \\ \Pi ::= \bullet\Gamma \end{cases}$$

- Interpretations:

General								$S \to G$		$G \to S$	
Φ		$\odot$		$<$		$>$		$\circ$		$\bullet$	
1	0	$\cdot$			$(/)$		$(\backslash)$	$\Box$	$\Box$	$\blacklozenge$	$\blacksquare$

Some specific structural rules

$$\text{one}\ \frac{}{\Phi \vdash \circ\Pi} \qquad \frac{\Gamma \vdash \circ\Pi \qquad \Delta \vdash \circ\Pi}{\Gamma \odot \Delta \vdash \circ\Pi}\ \text{abs}$$

$$\text{b-bal}\ \frac{\Pi \vdash \Sigma}{\bullet \circ \Pi \vdash \bullet \circ \Sigma} \qquad \frac{\Pi \vdash \Xi}{\circ\Pi \vdash \circ\Xi}\ \text{w-bal}$$

$$\omega\ \frac{(\Gamma^{(n)} \vdash \Delta \mid n \geq 1)}{\circ \bullet \Gamma \vdash \Delta} \qquad \frac{\circ\Pi \odot \circ\Pi \vdash \Delta}{\circ\Pi \vdash \Delta}\ \circ\text{-C}$$

Some derivations

- $\square\blacklozenge\alpha \cdot \square\blacklozenge\alpha \vdash \square\blacklozenge\alpha$

$$
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$$

- $\square\blacklozenge\alpha \vdash \square\blacklozenge\alpha \cdot \square\blacklozenge\alpha$

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\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha}{\bullet\alpha \vdash \blacklozenge\alpha}}{\blacklozenge\alpha \vdash \blacklozenge\alpha}}{\circ\blacklozenge\alpha \vdash \circ\blacklozenge\alpha}\ \text{w-bal}}{\circ\blacklozenge\alpha \vdash \square\blacklozenge\alpha} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha}{\bullet\alpha \vdash \blacklozenge\alpha}}{\blacklozenge\alpha \vdash \blacklozenge\alpha}}{\circ\blacklozenge\alpha \vdash \circ\blacklozenge\alpha}\ \text{w-bal}}{\circ\blacklozenge\alpha \vdash \square\blacklozenge\alpha}}{\dfrac{\circ\blacklozenge\alpha \odot \circ\blacklozenge\alpha \vdash \square\blacklozenge\alpha \cdot \square\blacklozenge\alpha}{\circ\blacklozenge\alpha \vdash \square\blacklozenge\alpha \cdot \square\blacklozenge\alpha}\ \circ\text{-C}}}{\square\blacklozenge\alpha \vdash \square\blacklozenge\alpha \cdot \square\blacklozenge\alpha}
$$

Consequences

Theorem (Soundness)

The rules in D.MKL is sound w.r.t. the class of HMKA.

Theorem (Completeness)

D.MKL is complete with respect to the class of MKA.

Theorem (Conservativity)

D.MKL is a conservative extension of H.MKL.

Theorem (Cut elimination)

D.MKL is cut-eliminable.

Theorem (Subformula property)

D.MKL has subformula property.

Thank you for your attention!