# Propositions as Sessions

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# Kohei Honda, 1959–2012

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# From Data Types to Session Types: A Basis for Concurrency and Distribution (ABCD)

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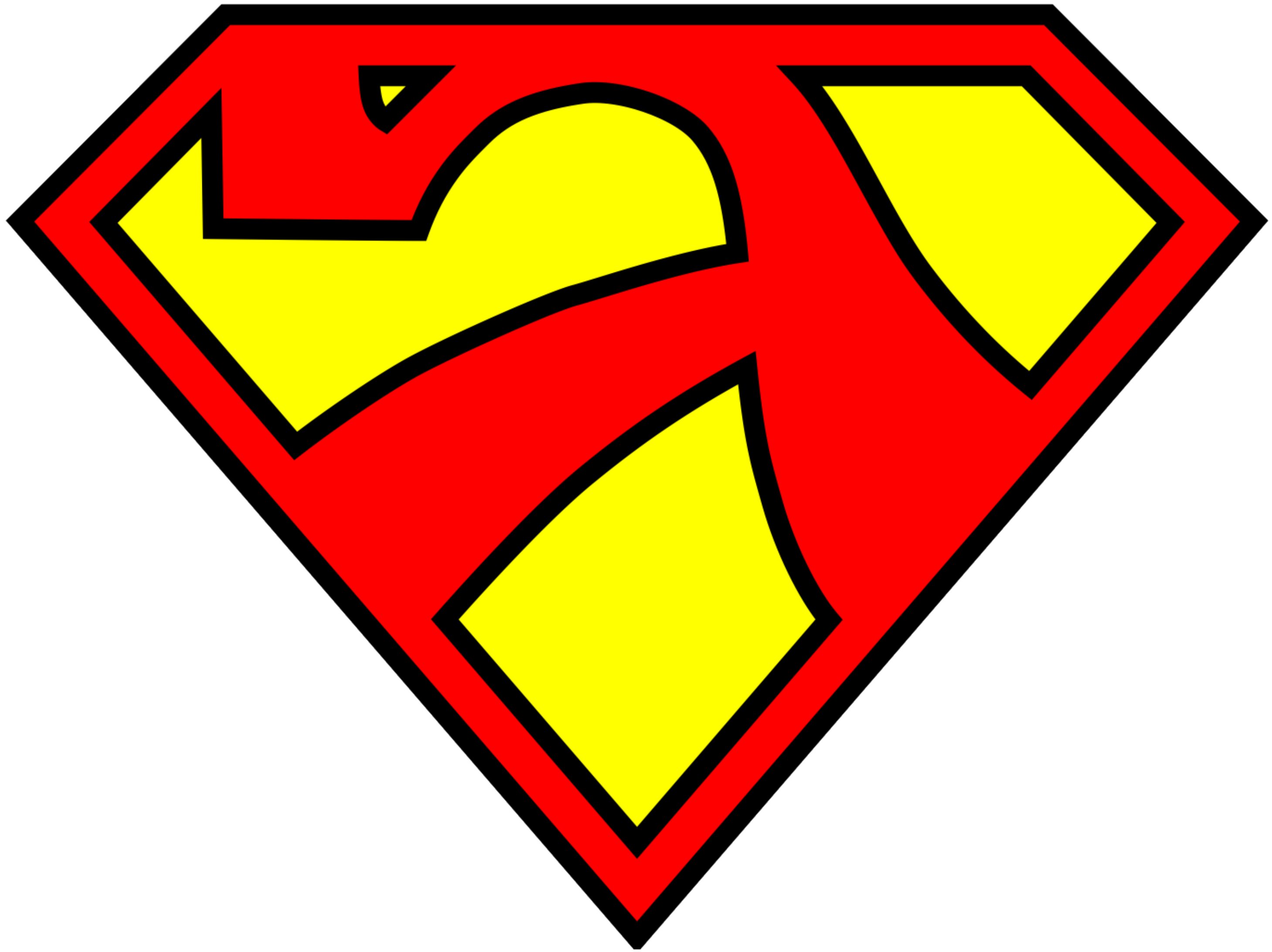

# Propositions as Types

| | | |
|---|---|---|
| propositions | *as* | types |
| proofs | *as* | programs |
| normalisation of proofs | *as* | evaluation of programs |

# Propositions as Types is robust

| | | |
|---|---|---|
| propositions | *as* | types |
| proofs | *as* | programs |
| normalisation of proofs | *as* | evaluation of programs |

| | | |
|---|---|---|
| Intuitionistic Natural Deduction | $\leftrightarrow$ | Simply-Typed Lambda Calculus |
| Quantification over propositions | $\leftrightarrow$ | Polymorphism |
| Quantification over individuals | $\leftrightarrow$ | Dependent types |
| Modal Logic | $\leftrightarrow$ | Monads (state, exceptions) |
| Classical-Intuitionistic Embedding | $\leftrightarrow$ | Continuation Passing Style |

# . . . but there's a missing link

| | | |
|---|---|---|
| propositions | *as* | types |
| proofs | *as* | programs |
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| | | |
|---|---|---|
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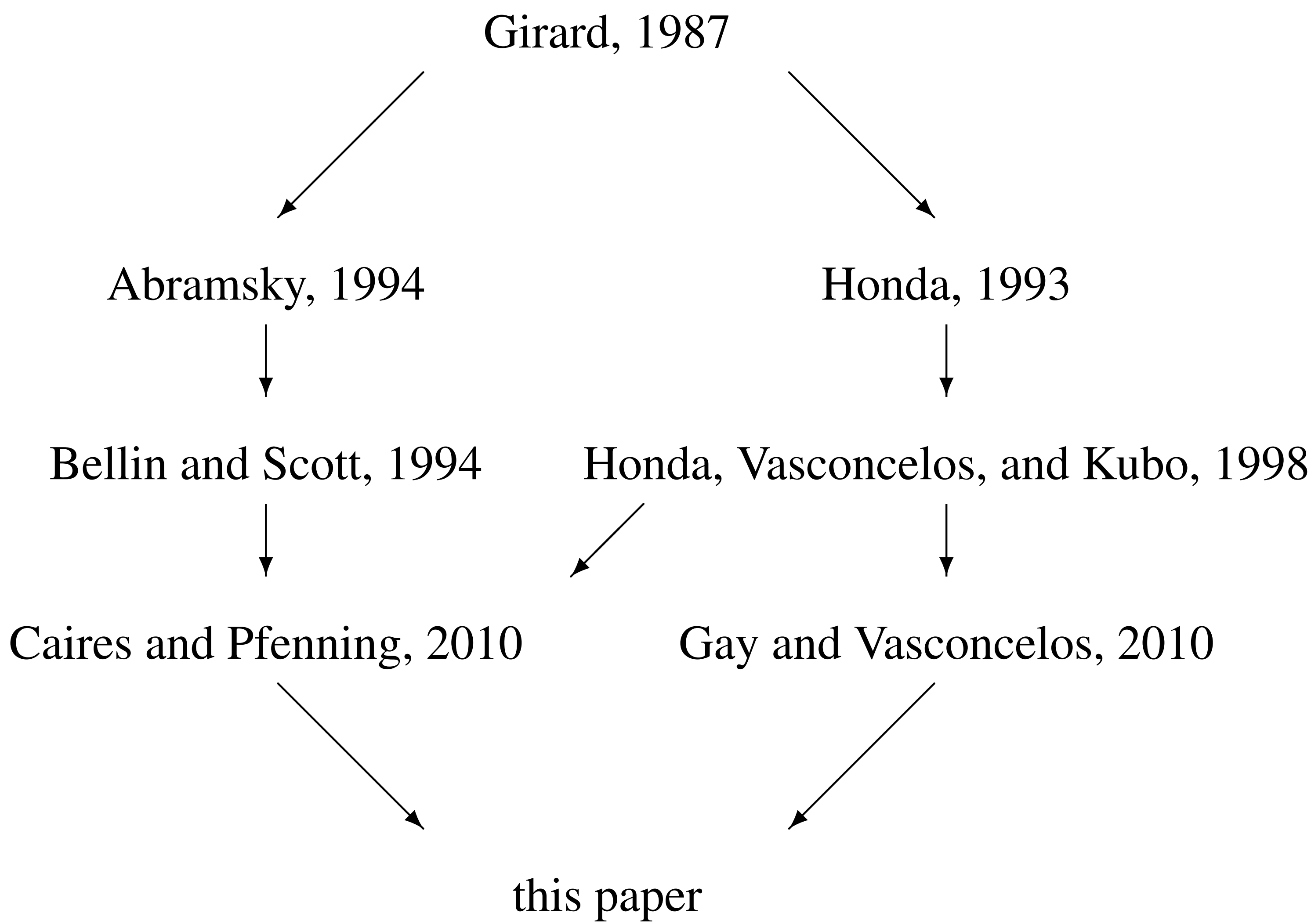
| | | |
| ??? | $\leftrightarrow$ | Process Calculus |

# Propositions as Sessions

| | | |
|---:|:---:|:---|
| propositions | *as* | types |
| proofs | *as* | programs |
| normalisation of proofs | *as* | evaluation of programs |

| | | |
|---:|:---:|:---|
| propositions | *as* | session types |
| proofs | *as* | processes |
| cut elimination | *as* | communication |

# Lines of development



# The Twist

- Abramsky, 1994: Proofs as Processes

$$\frac{P \vdash \Gamma, y : A \qquad Q \vdash \Delta, z : B}{\nu y, z.\, x\langle y, z\rangle.(P \mid Q) \vdash \Gamma, \Delta, x : A \otimes B}\ \otimes$$

$$\frac{R \vdash \Theta,\ y : A,\ z : B}{x(y, z).R \vdash \Theta,\ x : A \mathbin{\bindnasrepma} B}\ \mathbin{\bindnasrepma}$$

- this paper: Propositions as Sessions

$$\frac{P \vdash \Gamma,\ y : A \qquad Q \vdash \Delta,\ x : B}{\nu y.\, x\langle y\rangle.(P \mid Q) \vdash \Gamma,\ \Delta,\ x : A \otimes B}\ \otimes$$

$$\frac{R \vdash \Theta,\ y : A,\ x : B}{x(y).R \vdash \Theta,\ x : A \mathbin{\bindnasrepma} B}\ \mathbin{\bindnasrepma}$$

# A small change in notation

- With $\nu x.\, x\langle y\rangle$

$$\frac{P \vdash \Gamma, y : A \qquad Q \vdash \Delta, x : B}{\nu y.\, x\langle y\rangle.(P \mid Q) \vdash \Gamma, \Delta, x : A \otimes B}\otimes$$

- With $x[y]$

$$\frac{P \vdash \Gamma, y : A \qquad Q \vdash \Delta, x : B}{x[y].(P \mid Q) \vdash \Gamma, \Delta, x : A \otimes B}\otimes$$

# ILL vs. CLL

- Caires and Pfenning, 2010: Intuitionistic Linear Logic

$$\frac{\Gamma;\ \Delta \vdash P :: y : A \qquad \Gamma;\ \Delta' \vdash Q :: x : B}{\Gamma;\ \Delta, \Delta' \vdash \nu y.\, x\langle y\rangle.(P \mid Q) :: x : A \otimes B}\ \otimes\text{-R}$$

$$\frac{\Gamma;\ \Delta \vdash P :: y : A \quad \Gamma;\ \Delta', x : B \vdash Q :: z : C}{\Gamma;\ \Delta, \Delta', x : A \multimap B \vdash \nu y.x\langle y\rangle.(P \mid Q) :: z : C}\ \multimap\text{-L}$$

$$\frac{\Gamma;\ \Delta, y : A \vdash R :: x : B}{\Gamma;\ \Delta \vdash x(y).R :: x : A \multimap B}\ \multimap\text{-R} \qquad \frac{\Gamma;\ \Delta, y : A, x : B \vdash R :: z : C}{\Gamma;\ \Delta, x : A \otimes B \vdash x(y).R :: z : C}\ \otimes\text{-L}$$

- this paper: Classical Linear Logic

$$\frac{P \vdash \Gamma, y : A \qquad Q \vdash \Delta, x : B}{x[y].(P \mid Q) \vdash \Gamma, \Delta, x : A \otimes B}\ \otimes \qquad \frac{R \vdash \Theta, y : A, x : B}{x(y).R \vdash \Theta, x : A \parr B}\ \parr$$

# Axiom and Polymorphism

- Abramsky, 1994

$$\frac{}{x(z).w\langle z\rangle.0 \vdash w : X^{\perp},\ x : X}\ \textsf{Ax}$$

- Caires and Pfenning, 2010

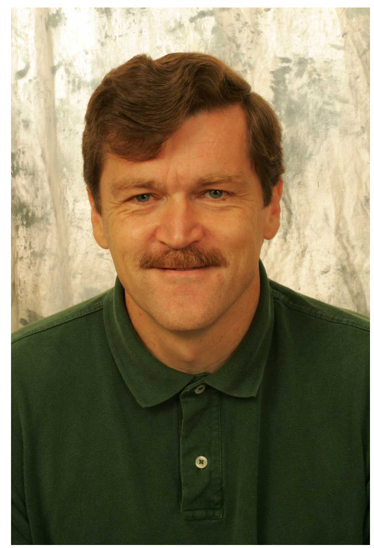
(no axiom)

- this paper (based on an idea from Caires and Pfenning, 2011)

$$\frac{}{w{\leftrightarrow}x \vdash w : A^{\perp},\ x : A}\ \textsf{Ax}$$

# Part I

# CP
# Classical Processes
# Caires-Pfenning

# Cut Elimination

## Theorem

(Subject Reduction)
If $P \vdash \Gamma$ and $P \Longrightarrow Q$ then $Q \vdash \Gamma$.

(Cut Elimination)
If $P \vdash \Gamma$ then there exists a $Q$
such that $P \Longrightarrow^* Q$ and $Q$ is not a $\mathsf{Cut}$.

# Types

$A, B, C ::=$

| | | | |
|---|---|---|---|
| $X$ | type variable | $X^\perp$ | dual of type variable |
| $A \otimes B$ | output $A$ then behave as $B$ | $A \parr B$ | input $A$ then behave as $B$ |
| $A \oplus B$ | select from $A$ or $B$ | $A \,\&\, B$ | offer choice of $A$ or $B$ |
| $!A$ | replicated input | $?A$ | replicated output |
| $\exists X.B$ | output a type | $\forall X.B$ | input a type |
| $1$ | unit for $\otimes$ | $\perp$ | unit for $\parr$ |
| $0$ | unit for $\oplus$ | $\top$ | unit for $\&$ |

# Duals

$$(X)^\perp = X^\perp \qquad (X^\perp)^\perp = X$$

$$(A \otimes B)^\perp = A^\perp ⅋ B^\perp \qquad (A ⅋ B)^\perp = A^\perp \otimes B^\perp$$

$$(A \oplus B)^\perp = A^\perp \,\&\, B^\perp \qquad (A \,\&\, B)^\perp = A^\perp \oplus B^\perp$$

$$(!A)^\perp = ?A^\perp \qquad (?A)^\perp = !A^\perp$$

$$(\exists X.B)^\perp = \forall X.B^\perp \qquad (\forall X.B)^\perp = \exists X.B^\perp$$

$$1^\perp = \perp \qquad \perp^\perp = 1$$

$$0^\perp = \top \qquad \top^\perp = 0$$

# Processes

$P, Q, R ::=$

| | | | |
|---|---|---|---|
| $x \leftrightarrow y$ | link | $\nu x : A.(P \mid Q)$ | parallel composition |
| $x[y].(P \mid Q)$ | output | $x(y).P$ | input |
| $x[\mathsf{inl}].P$ | left selection | $x.\mathsf{case}(P, Q)$ | choice |
| $x[\mathsf{inr}].P$ | right selection | | |
| $?x[y].P$ | replicated output | $!x(y).P$ | replicated input |
| $x[A].P$ | send type | $x(X).P$ | receive type |
| $x[\,].0$ | empty output | $x(\,).P$ | empty input |
| | | $x.\mathsf{case}()$ | empty choice |

Forms $x(y).P$ and $!x(y).P$ behave like the same forms in $\pi$-calculus.

Forms $x[y].P$ and $?x[y].P$ behave like form $\nu y.\, x\langle y\rangle.P$ in $\pi$-calculus.

# Structural rules

$$\dfrac{}{w{\leftrightarrow}x \vdash w : A^{\perp},\, x : A}\ \textsf{Ax}$$

$$\dfrac{P \vdash \Gamma,\, x : A \qquad Q \vdash \Delta,\, x : A^{\perp}}{\nu x : A.(P \mid Q) \vdash \Gamma,\, \Delta}\ \textsf{Cut}$$

(AxCut)

$$\dfrac{\dfrac{}{w{\leftrightarrow}x \vdash w : A^{\perp},\, x : A}\ \textsf{Ax} \qquad P \vdash \Gamma,\, x : A^{\perp}}{\nu x.(w{\leftrightarrow}x \mid P) \vdash \Gamma,\, w : A^{\perp}}\ \textsf{Cut} \quad \Longrightarrow \quad P\{w/x\} \vdash \Gamma,\, w : A^{\perp}$$

# Structural rules—equivalences

(Swap)

$$\frac{P \vdash \Gamma, x : A \quad Q \vdash \Delta, x : A^\perp}{\nu x : A.(P \mid Q) \vdash \Gamma, \Delta}\ \mathsf{Cut} \quad \equiv \quad \frac{Q \vdash \Delta, x : A^\perp \quad P \vdash \Gamma, x : A}{\nu x : A^\perp.(Q \mid P) \vdash \Gamma, \Delta}\ \mathsf{Cut}$$

(Assoc)

$$\frac{\dfrac{P \vdash \Gamma, x : A \quad Q \vdash \Delta, x : A^\perp, y : B}{\nu x.(P \mid Q) \vdash \Gamma, \Delta, y : B}\ \mathsf{Cut} \quad R \vdash \Theta, y : B^\perp}{\nu y.(\nu x.(P \mid Q) \mid R) \vdash \Gamma, \Delta, \Theta}\ \mathsf{Cut} \quad \equiv$$

$$\frac{P \vdash \Gamma, x : A \quad \dfrac{Q \vdash \Delta, x : A^\perp, y : B \quad R \vdash \Theta, y : B^\perp}{\nu y.(Q \mid R) \vdash \Delta, \Theta, x : A^\perp}\ \mathsf{Cut}}{\nu x.(P \mid \nu y.(Q \mid R)) \vdash \Gamma, \Delta, \Theta}\ \mathsf{Cut}$$

# Input and Output—Multiplicatives

$$\frac{P \vdash \Gamma, y : A \quad Q \vdash \Delta, x : B}{x[y].(P \mid Q) \vdash \Gamma, \Delta, x : A \otimes B} \otimes$$

$$\frac{R \vdash \Theta, y : A, x : B}{x(y).R \vdash \Theta, x : A \parr B} \parr$$

$(\beta_{\otimes\parr})$

$$\frac{\dfrac{P \vdash \Gamma, y : A \quad Q \vdash \Delta, x : B}{x[y].(P \mid Q) \vdash \Gamma, \Delta, x : A \otimes B} \otimes \quad \dfrac{R \vdash \Theta, y : A^\perp, x : B^\perp}{x(y).R \vdash \Theta, x : A^\perp \parr B^\perp} \parr}{\nu x.(x[y].(P \mid Q) \mid x(y).R) \vdash \Gamma, \Delta, \Theta} \text{Cut} \quad \Longrightarrow$$

$$\frac{P \vdash \Gamma, y : A \quad \dfrac{Q \vdash \Delta, x : B \quad R \vdash \Theta, y : A^\perp, x : B^\perp}{\nu x.(Q \mid R) \vdash \Delta, \Theta, y : A^\perp} \text{Cut}}{\nu y.(P \mid \nu x.(Q \mid R)) \vdash \Gamma, \Delta, \Theta} \text{Cut}$$

# Input and Output—identity, swap

$$\dfrac{\dfrac{\dfrac{}{y{\leftrightarrow}x \vdash y : A^\perp,\ x : A}\ \mathsf{Ax} \qquad \dfrac{}{w{\leftrightarrow}z \vdash w : B^\perp,\ z : B}\ \mathsf{Ax}}{z[x].(w{\leftrightarrow}z \mid y{\leftrightarrow}x) \vdash y : A^\perp,\ w : B^\perp,\ z : A \otimes B}\ \otimes}{w(y).z[x].(w{\leftrightarrow}z \mid y{\leftrightarrow}x) \vdash w : A^\perp \parr B^\perp,\ z : A \otimes B}\ \parr$$

$$\dfrac{\dfrac{\dfrac{}{w{\leftrightarrow}z \vdash w : B^\perp,\ z : B}\ \mathsf{Ax} \qquad \dfrac{}{y{\leftrightarrow}x \vdash y : A^\perp,\ x : A}\ \mathsf{Ax}}{x[z].(w{\leftrightarrow}z \mid y{\leftrightarrow}x) \vdash w : B^\perp,\ y : A^\perp,\ x : B \otimes A}\ \otimes}{w(y).x[z].(w{\leftrightarrow}z \mid y{\leftrightarrow}x) \vdash w : A^\perp \parr B^\perp,\ x : B \otimes A}\ \parr$$

# Selection and Choice—Additives

$$\frac{P \vdash \Gamma,\ x : A}{x[\mathsf{inl}].P \vdash \Gamma,\ x : A \oplus B}\ \oplus_1 \qquad \frac{P \vdash \Gamma,\ x : B}{x[\mathsf{inr}].P \vdash \Gamma,\ x : A \oplus B}\ \oplus_2$$

$$\frac{Q \vdash \Delta,\ x : A \quad R \vdash \Delta,\ x : B}{x.\mathsf{case}(Q, R) \vdash \Delta,\ x : A \,\&\, B}\ \&$$

$(\beta_{\oplus\&})$

$$\frac{\dfrac{P \vdash \Gamma,\ x : A}{x[\mathsf{inl}].P \vdash \Gamma,\ x : A \oplus B}\ \oplus_1 \quad \dfrac{Q \vdash \Delta,\ x : A^\perp \quad R \vdash \Delta,\ x : B^\perp}{x.\mathsf{case}(Q, R) \vdash \Delta,\ x : A^\perp \,\&\, B^\perp}\ \&}{\nu x.(x[\mathsf{inl}].P \mid x.\mathsf{case}(Q, R)) \vdash \Gamma,\ \Delta}\ \mathsf{Cut} \quad \Longrightarrow$$

$$\frac{P \vdash \Gamma,\ x : A \quad Q \vdash \Delta,\ x : A^\perp}{\nu x.(P \mid Q) \vdash \Gamma,\ \Delta}\ \mathsf{Cut}$$

# Servers and Clients—Exponentials

$$\frac{P \vdash ?\Gamma, y : A}{!x(y).P \vdash ?\Gamma, x : !A}\ !$$

$$\frac{Q \vdash \Delta, y : A}{?x[y].Q \vdash \Delta, x : ?A}\ ?$$

$(\beta_{!?})$

$$\dfrac{\dfrac{P \vdash ?\Gamma, y : A}{!x(y).P \vdash ?\Gamma, x : !A}\ ! \qquad \dfrac{Q \vdash \Delta, y : A^{\perp}}{?x[y].Q \vdash \Delta, x : ?A^{\perp}}\ ?}{\nu x.(!x(y).P \mid ?x[y].Q) \vdash ?\Gamma, \Delta}\ \mathsf{Cut} \qquad \Longrightarrow$$

$$\frac{P \vdash ?\Gamma, y : A \quad Q \vdash \Delta, y : A^{\perp}}{\nu y.(P \mid Q) \vdash ?\Gamma, \Delta}\ \mathsf{Cut}$$

# Weakening and Contraction

$$\frac{Q \vdash \Delta}{Q \vdash \Delta,\ x : ?A}\ \textsf{Weaken}$$

$$\frac{Q \vdash \Delta,\ x : ?A,\ x' : ?A}{Q\{x/x'\} \vdash \Delta,\ x : ?A}\ \textsf{Contract}$$

$(\beta_{!W})$

$$\frac{\dfrac{P \vdash ?\Gamma,\ y : A}{!x(y).P \vdash ?\Gamma,\ x : !A}\ ! \qquad \dfrac{Q \vdash \Delta}{Q \vdash \Delta,\ x : ?A^{\perp}}\ \textsf{Weaken}}{\nu x.(!x(y).P \mid Q) \vdash ?\Gamma,\ \Delta}\ \textsf{Cut} \quad \Longrightarrow$$

$$\frac{Q \vdash \Delta}{\overline{Q \vdash ?\Gamma,\ \Delta}}\ \textsf{Weaken}$$

# Weakening and Contraction, continued

$(\beta_{!C})$

$$\dfrac{\dfrac{P \vdash ?\Gamma,\, y : A}{!x(y).P \vdash ?\Gamma,\, x : !A}\;! \qquad \dfrac{Q \vdash \Delta,\, x : ?A,\, x' : ?A}{Q\{x/x'\} \vdash \Delta,\, x : ?A}\;\textsf{Contract}}{\nu x.(!x(y).P \mid Q\{x/x'\}) \vdash ?\Gamma,\, \Delta}\;\textsf{Cut} \qquad \Longrightarrow$$

$$\dfrac{\dfrac{P \vdash ?\Gamma,\, y : A}{!x(y).P \vdash ?\Gamma,\, x : !A}\;! \qquad \dfrac{\dfrac{P' \vdash ?\Gamma',\, y' : A}{!x'(y').P' \vdash ?\Gamma',\, x' : !A}\;! \qquad Q \vdash \Delta,\, x : ?A^{\perp},\, x' : ?A^{\perp}}{\nu x'.(!x'(y').P' \mid Q) \vdash ?\Gamma',\, \Delta,\, x : ?A^{\perp}}\;\textsf{Cut}}{\nu x.(!x(y).P \mid \nu x'.(!x'(y').P' \mid Q)) \vdash ?\Gamma,\, ?\Gamma',\, \Delta}\;\textsf{Cut}$$

$$\begin{array}{c} \nu x.(!x(y).P \mid \nu x'.(!x'(y').P' \mid Q)) \vdash ?\Gamma,\, ?\Gamma',\, \Delta \\ \hline\hline \nu x.(!x(y).P \mid \nu x'.(!x'(y).P \mid Q)) \vdash ?\Gamma,\, \Delta \end{array}\;\textsf{Contract}$$

# Polymorphism—Quantifiers

$$\frac{P \vdash \Gamma,\, x : B\{A/X\}}{x[A].P \vdash \Gamma,\, x : \exists X.B}\ \exists$$

$$\frac{Q \vdash \Delta,\, x : B}{x(X).Q \vdash \Delta,\, x : \forall X.B}\ \forall \quad (X \notin \mathsf{fv}(\Delta))$$

$(\beta_{\exists\forall})$

$$\frac{\dfrac{P \vdash \Gamma,\, x : B\{A/X\}}{x[A].P \vdash \Gamma,\, x : \exists X.B}\ \exists \qquad \dfrac{Q \vdash \Delta,\, x : B^{\perp}}{x(X).Q \vdash \Delta,\, x : \forall X.B^{\perp}}\ \forall}{\nu x.(x[A].P \mid x(X).Q) \vdash \Gamma,\, \Delta}\ \mathsf{Cut} \quad \Longrightarrow$$

$$\frac{P \vdash \Gamma,\, x : B\{A/X\} \qquad Q\{A/X\} \vdash \Delta,\, x : B^{\perp}\{A/X\}}{\nu x.(P \mid Q\{A/X\}) \vdash \Gamma,\, \Delta}\ \mathsf{Cut}$$

# Units

$$\dfrac{}{x[\,].0 \vdash x : 1}\ 1 \qquad \dfrac{P \vdash \Gamma}{x().P \vdash \Gamma,\ x : \bot}\ \bot$$

(no rule for 0) $$\dfrac{}{x.\mathsf{case}() \vdash \Gamma,\ x : \top}\ \top$$

$(\beta_{1\bot})$

$$\dfrac{\dfrac{}{x[\,].0 \vdash x : 1}\ 1 \quad \dfrac{P \vdash \Gamma}{x().P \vdash \Gamma,\ x : \bot}\ \bot}{\nu x.(x[\,].0 \mid x().P) \vdash \Gamma}\ \mathsf{Cut} \quad \Longrightarrow \quad P \vdash \Gamma$$

$(\beta_{0\top})$

(no rule for 0 with $\top$)

# Commuting Conversions

$(\kappa_{\otimes 1})$

$$\dfrac{\dfrac{P \vdash \Gamma,\, y : A,\, z : C \qquad Q \vdash \Delta,\, x : B}{x[y].(P \mid Q) \vdash \Gamma,\, \Delta,\, x : A \otimes B}\,\otimes \qquad R \vdash \Theta,\, z : C^{\perp}}{\nu z.(x[y].(P \mid Q) \mid R) \vdash \Gamma,\, \Delta,\, \Theta,\, x : A \otimes B}\,\text{Cut} \quad \Longrightarrow$$

$$\dfrac{\dfrac{P \vdash \Gamma,\, y : A,\, z : C \qquad R \vdash \Theta,\, z : C^{\perp}}{\nu z.(P \mid R) \vdash \Gamma,\, \Theta,\, y : A}\,\text{Cut} \qquad Q \vdash \Delta,\, x : B}{x[y].(\nu z.(P \mid R) \mid Q) \vdash \Gamma,\, \Delta,\, \Theta,\, x : A \otimes B}\,\otimes$$

$(\kappa_{\otimes 2})$

$$\dfrac{\dfrac{P \vdash \Gamma,\, y : A \qquad Q \vdash \Delta,\, x : B,\, z : C}{x[y].(P \mid Q) \vdash \Gamma,\, \Delta,\, x : A \otimes B}\,\otimes \qquad R \vdash \Theta,\, z : C^{\perp}}{\nu z.(x[y].(P \mid Q) \mid R) \vdash \Gamma,\, \Delta,\, \Theta,\, x : A \otimes B}\,\text{Cut} \quad \Longrightarrow$$

$$\dfrac{P \vdash \Gamma,\, y : A \qquad \dfrac{Q \vdash \Delta,\, x : B,\, z : C \qquad R \vdash \Theta,\, z : C^{\perp}}{\nu z.(Q \mid R) \vdash \Delta,\, \Theta,\, x : B}\,\text{Cut}}{x[y].(P \mid \nu z.(Q \mid R)) \vdash \Gamma,\, \Delta,\, \Theta,\, x : A \otimes B}\,\otimes$$

# Commuting Conversions

$$(\kappa_{\otimes 1}) \quad \nu z.(x[y].(P \mid Q) \mid R) \Longrightarrow x[y].(\nu z.(P \mid R) \mid Q), \quad \text{if } z \in \mathsf{fn}(P)$$

$$(\kappa_{\otimes 2}) \quad \nu z.(x[y].(P \mid Q) \mid R) \Longrightarrow x[y].(P \mid \nu z.(Q \mid R)), \quad \text{if } z \in \mathsf{fn}(Q)$$

$$(\kappa_{\invamp}) \quad \nu z.(x(y).P \mid Q) \Longrightarrow x(y).\nu z.(P \mid Q)$$

$$(\kappa_{\oplus}) \quad \nu z.(x[\mathsf{inl}].P \mid Q) \Longrightarrow x[\mathsf{inl}].\nu z.(P \mid Q)$$

$$(\kappa_{\&}) \quad \nu z.(x.\mathsf{case}(P, Q) \mid R) \Longrightarrow x.\mathsf{case}(\nu z.(P \mid R), \nu z.(Q \mid R))$$

$$(\kappa_{!}) \quad \nu z.(!x(y).P \mid Q) \Longrightarrow !x(y).\nu z.(P \mid Q)$$

$$(\kappa_{?}) \quad \nu z.(?x[y].P \mid Q) \Longrightarrow ?x[y].\nu z.(P \mid Q)$$

$$(\kappa_{\exists}) \quad \nu z.(x[A].P \mid Q) \Longrightarrow x[A].\nu z.(P \mid Q)$$

$$(\kappa_{\forall}) \quad \nu z.(x(X).P \mid Q) \Longrightarrow x(X).\nu z.(P \mid Q)$$

$$(\kappa_{\bot}) \quad \nu z.(x().P \mid Q) \Longrightarrow x().\nu z.(P \mid Q)$$

$$(\kappa_{\top}) \quad \nu z.(x.\mathsf{case}() \mid Q) \Longrightarrow x.\mathsf{case}()$$

# No congruence!

If our goal was to eliminate all cuts, we would need to introduce congruence rules, such as

$$\frac{P \Longrightarrow Q}{x(y).P \Longrightarrow x(y).Q}$$

and similarly for each operator. Such rules do not correspond well to our notion of computation on processes, so we omit them; this is analogous to the usual practice of not permitting reduction under lambda.

# Cut Elimination

## **Theorem**

(Subject Reduction)

If $P \vdash \Gamma$ and $P \Longrightarrow Q$ then $Q \vdash \Gamma$.

(Cut Elimination)

If $P \vdash \Gamma$ then there exists a $Q$

such that $P \Longrightarrow^* Q$ and $Q$ is not a $\mathsf{Cut}$.

# Part II

# GV

# Good Variation

# Gay-Vasconcelos

# Type Preservation

## Theorem

(Translation preserves types)

If $\Phi \vdash M : T$

then $[\![M]\!]z \vdash [\![\Phi]\!]^{\perp}, z : [\![T]\!]$.

# Session Types

$S ::=$

| | |
|---|---|
| $!T.S$ | output value of type $T$ then behave as $S$ |
| $?T.S$ | input value of type $T$ then behave as $S$ |
| $\oplus\{l_i : S_i\}_{i\in I}$ | select from behaviours $S_i$ with label $l_i$ |
| $\&\{l_i : S_i\}_{i\in I}$ | offer choice of behaviours $S_i$ with label $l_i$ |
| $\mathsf{end}_!$ | terminator, convenient for use with output |
| $\mathsf{end}_?$ | terminator, convenient for use with input |

Each session $S$ has a dual $\overline{S}$:

$$
\begin{array}{rclcrcl}
\overline{!T.S} & = & ?T.\overline{S} & \quad & \overline{?T.S.} & = & !T.\overline{S} \\
\overline{\oplus(l_i : S_i)_{i\in I}} & = & \&(l_i : \overline{S}_i)_{i\in I} & & \overline{\&(l_i : S_i)_{i\in I}} & = & \oplus(l_i : \overline{S}_i)_{i\in I} \\
\overline{\mathsf{end}_!} & = & \mathsf{end}_? & & \overline{\mathsf{end}_?} & = & \mathsf{end}_!
\end{array}
$$

# Types

$$T, U, V ::=$$

| | |
|---|---|
| $S$ | session (linear) |
| $T \otimes U$ | tensor product (linear) |
| $T \multimap U$ | function (linear) |
| $T \rightarrow U$ | function (unlimited) |
| $\mathsf{Unit}$ | unit (unlimited) |

Each type is classified as linear or unlimited:

$$\mathsf{lin}(S) \quad \mathsf{lin}(T \otimes U) \quad \mathsf{lin}(T \multimap U)$$

$$\mathsf{un}(T \rightarrow U) \quad \mathsf{un}(\mathsf{Unit})$$

# Terms

$L, M, N ::=$

| | |
|---|---|
| $x$ | identifier |
| unit | unit constant |
| $\lambda x.\, N$ | function abstraction |
| $L\ M$ | function application |
| $(M, N)$ | pair construction |
| let $(x, y) = M$ in $N$ | pair deconstruction |
| send $M\ N$ | send value $M$ on channel $N$ |
| receive $M$ | receive from channel $M$ |
| select $l\ M$ | select label $l$ on channel $M$ |
| case $M$ of $\{l_i : x.N_i\}_{i \in I}$ | offer choice on channel $M$ |
| with $x$ connect $M$ to $N$ | connect $M$ to $N$ by channel $x$ |
| terminate $M$ | terminate input |

# Functions and Pairs

$$\frac{}{x : T \vdash x : T}\ \textsf{Id} \qquad \frac{}{\vdash \textsf{unit} : \textsf{Unit}}\ \textsf{Unit}$$

$$\frac{\Phi \vdash N : U \quad \textsf{un}(T)}{\Phi,\ x : T \vdash N : U}\ \textsf{Weaken} \qquad \frac{\Phi,\ x : T,\ x' : T \vdash N : U \quad \textsf{un}(T)}{\Phi,\ x : T \vdash N\{x/x'\} : U}\ \textsf{Contract}$$

$$\frac{\Phi,\ x : T \vdash N : U}{\Phi \vdash \lambda x.\, N : T \multimap U}\ \multimap\textsf{-I} \qquad \frac{\Phi \vdash L : T \multimap U \quad \Psi \vdash M : T}{\Phi,\ \Psi \vdash L\, M : U}\ \multimap\textsf{-E}$$

$$\frac{\Phi \vdash L : T \multimap U \quad \textsf{un}(\Phi)}{\Phi \vdash L : T \to U}\ \to\textsf{-I} \qquad \frac{\Phi \vdash L : T \to U}{\Phi \vdash L : T \multimap U}\ \to\textsf{-E}$$

$$\frac{\Phi \vdash M : T \quad \Psi \vdash N : U}{\Phi,\ \Psi \vdash (M, N) : T \otimes U}\ \otimes\textsf{-I} \qquad \frac{\Phi \vdash M : T \otimes U \quad \Psi,\ x : T,\ y : U \vdash N : V}{\Phi,\ \Psi \vdash \textsf{let}\ (x, y) = M\ \textsf{in}\ N : V}\ \otimes\textsf{-E}$$

# Communication

$$\frac{\Phi \vdash M : T \quad \Psi \vdash N : !T.S}{\Phi, \Psi \vdash \textsf{send}\ M\ N : S}\ \textsf{Send} \qquad \frac{\Phi \vdash M : ?T.S}{\Phi \vdash \textsf{receive}\ M : T \otimes S}\ \textsf{Receive}$$

$$\frac{\Phi \vdash M : \oplus\{l_i : S_i\}_{i \in I}}{\Phi \vdash \textsf{select}\ l_j\ M : S_j}\ \textsf{Select}$$

$$\frac{\Phi \vdash M : \&\{l_i : S_i\}_{i \in I} \quad (\Psi, x : S_i \vdash N_i : T)_{i \in I}}{\Phi, \Psi \vdash \textsf{case}\ M\ \textsf{of}\ \{l_i : x.N_i\}_{i \in I} : T}\ \textsf{Case}$$

$$\frac{\Phi, x : S \vdash M : \textsf{end}_! \quad \Psi, x : \overline{S} \vdash N : T}{\Phi, \Psi \vdash \textsf{with}\ x\ \textsf{connect}\ M\ \textsf{to}\ N : T}\ \textsf{Connect}$$

$$\frac{\Phi \vdash M : T \otimes \textsf{end}_?}{\Phi \vdash \textsf{terminate}\ M : T}\ \textsf{Terminate}$$

# Translation of Sessions

$$
\begin{aligned}
[\![ !T.S ]\!] &= [\![ T ]\!]^{\perp} \parr [\![ S ]\!] \\
[\![ ?T.S ]\!] &= [\![ T ]\!] \otimes [\![ S ]\!] \\
[\![ \oplus\{l_i : S_i\}_{i \in I} ]\!] &= [\![ S_1 ]\!] \,\&\, \cdots \,\&\, [\![ S_n ]\!], \quad I = \{1, \ldots, n\} \\
[\![ \&\{l_i : S_i\}_{i \in I} ]\!] &= [\![ S_1 ]\!] \oplus \cdots \oplus [\![ S_n ]\!], \quad I = \{1, \ldots, n\} \\
[\![ \mathsf{end}_! ]\!] &= \perp \\
[\![ \mathsf{end}_? ]\!] &= 1
\end{aligned}
$$

Translation preserves duality:

$$
[\![ \overline{S} ]\!] \quad = \quad [\![ S ]\!]^{\perp}
$$

# Translation of Types

$$[\![T \multimap U]\!] = [\![T]\!]^{\perp} \parr [\![U]\!]$$
$$[\![T \rightarrow U]\!] = !([\![T]\!]^{\perp} \parr [\![U]\!])$$
$$[\![T \otimes U]\!] = [\![T]\!] \otimes [\![U]\!]$$
$$[\![\mathsf{Unit}]\!] = !\top$$

An unlimited type translates to a type constructed with !:

$$\text{If } \mathsf{un}(T) \text{ then } [\![T]\!] = !A, \text{ for some } A.$$

# Translation of Linear Functions

$$\left[\!\!\left[ \frac{\Phi, x : T \vdash N : U}{\Phi \vdash \lambda x.\, N : T \multimap U} \multimap\text{-I} \right]\!\!\right] z \quad = \quad \frac{[\![N]\!]z \vdash [\![\Phi]\!]^\perp, x : [\![T]\!]^\perp, z : [\![U]\!]}{z(x).[\![N]\!]z \vdash [\![\Phi]\!]^\perp, z : [\![T]\!]^\perp \parr [\![U]\!]} \parr$$

$$\left[\!\!\left[ \frac{\Phi \vdash L : T \multimap U \qquad \Psi \vdash M : T}{\Phi, \Psi \vdash L\, M : U} \multimap\text{-E} \right]\!\!\right] z \quad =$$

$$\frac{[\![L]\!]y \vdash [\![\Phi]\!]^\perp, y : [\![T]\!]^\perp \parr [\![U]\!] \qquad \dfrac{[\![M]\!]x \vdash [\![\Psi]\!]^\perp, x : [\![T]\!] \qquad y{\leftrightarrow}z \vdash y : [\![U]\!]^\perp, z : [\![U]\!]}{y[x].([\![M]\!]x \mid y{\leftrightarrow}z) \vdash [\![\Psi]\!]^\perp, y : [\![T]\!] \otimes [\![U]\!]^\perp, z : [\![U]\!]} \otimes}{\nu y.([\![L]\!]y \mid y[x].([\![M]\!]x \mid y{\leftrightarrow}z)) \vdash [\![\Phi]\!]^\perp, [\![\Psi]\!]^\perp, z : [\![U]\!]} \text{Cut}$$

# Translation of Unlimited Functions

$$\left[\!\!\left[ \frac{\Phi \vdash L : T \multimap U \quad \mathsf{un}(\Phi)}{\Phi \vdash L : T \to U} \to\text{-}\mathsf{I} \right]\!\!\right] z \quad =$$

$$\frac{[\![L]\!]y \vdash [\![\Phi]\!]^\perp,\ y : [\![T \multimap U]\!]}{!z(y).[\![L]\!]y \vdash [\![\Phi]\!]^\perp,\ z : ![\![T \multimap U]\!]}\ !$$

$$\left[\!\!\left[ \frac{\Phi \vdash L : T \to U}{\Phi \vdash L : T \multimap U} \to\text{-}\mathsf{E} \right]\!\!\right] z \quad =$$

$$\dfrac{[\![L]\!]y \vdash [\![\Phi]\!]^\perp,\ y : ![\![T \multimap U]\!] \qquad \dfrac{\dfrac{}{x{\leftrightarrow}z \vdash x : [\![T \multimap U]\!]^\perp,\ z : [\![T \multimap U]\!]}\ \mathsf{Ax}}{?y[x].x{\leftrightarrow}z \vdash y : ?[\![T \multimap U]\!]^\perp,\ z : [\![T \multimap U]\!]}\ ?}{\nu y.([\![L]\!]y \mid ?y[x].x{\leftrightarrow}z) \vdash [\![\Phi]\!]^\perp,\ z : [\![T \multimap U]\!]}\ \mathsf{Cut}$$

# Translation of Send and Receive

$$\left[\!\!\left[ \frac{\Phi \vdash M : T \quad \Psi \vdash N : !T.S}{\Phi, \Psi \vdash \mathsf{send}\ M\ N : S}\ \mathsf{Send} \right]\!\!\right] z \quad =$$

$$\dfrac{\dfrac{[\![M]\!]y \vdash [\![\Phi]\!]^{\perp}, y : [\![T]\!] \quad x{\leftrightarrow}z \vdash x : [\![S]\!]^{\perp}, z : [\![S]\!]}{x[y].([\![M]\!]y \mid x{\leftrightarrow}z) \vdash [\![\Phi]\!]^{\perp}, x : [\![T]\!] \otimes [\![S]\!]^{\perp}}\otimes \qquad [\![N]\!]x \vdash [\![\Psi]\!]^{\perp}, x : [\![T]\!]^{\perp} \parr [\![S]\!]}{\nu x.(x[y].([\![M]\!]y \mid x{\leftrightarrow}z) \mid [\![N]\!]x) \vdash [\![\Phi]\!]^{\perp}, [\![\Psi]\!]^{\perp}, z : [\![S]\!]}\ \mathsf{Cut}$$

$$\left[\!\!\left[ \frac{\Phi \vdash M : ?T.S}{\Phi \vdash \mathsf{receive}\ M : T \otimes S}\ \mathsf{Receive} \right]\!\!\right] z \quad =$$

$$[\![M]\!]z \vdash [\![\Phi]\!], z : [\![T]\!] \otimes [\![S]\!]$$

# Translation of Connect and Terminate

$$\left[\!\!\left[ \frac{\Phi, x : S \vdash M : \mathsf{end}_! \quad \Psi, x : \overline{S} \vdash N : T}{\Phi, \Psi \vdash \mathsf{with}\ x\ \mathsf{connect}\ M\ \mathsf{to}\ N : T}\ \mathsf{Connect} \right]\!\!\right] z \quad =$$

$$\dfrac{\dfrac{[\![M]\!]y \vdash [\![\Phi]\!]^\perp, x : [\![S]\!]^\perp, y : \bot \qquad \dfrac{}{y[].0 \vdash y : \mathbf{1}}\ \mathsf{1}}{\nu y.([\![M]\!]y \mid y[].0) \vdash [\![\Phi]\!]^\perp, x : [\![S]\!]^\perp}\ \mathsf{Cut} \qquad [\![N]\!]z \vdash [\![\Psi]\!]^\perp, x : [\![S]\!], z : [\![T]\!]}{\nu x.(\nu y.([\![M]\!]y \mid y[].0) \mid [\![N]\!]z) \vdash [\![\Phi]\!]^\perp, [\![\Psi]\!]^\perp, z : [\![T]\!]}$$

$$\left[\!\!\left[ \frac{\Phi \vdash M : T \otimes \mathsf{end}_?}{\Phi \vdash \mathsf{terminate}\ M : T}\ \mathsf{Terminate} \right]\!\!\right] z \quad =$$

$$\dfrac{[\![M]\!]y \vdash [\![\Phi]\!]^\perp, y : [\![T]\!] \otimes \mathbf{1} \qquad \dfrac{\dfrac{z \leftrightarrow y \vdash z : [\![T]\!], y : [\![T]\!]^\perp}{x().z \leftrightarrow y \vdash z : [\![T]\!], y : [\![T]\!]^\perp, x : \bot}\ \bot}{y(x).x().z \leftrightarrow y \vdash z : [\![T]\!], y : [\![T]\!]^\perp \parr \bot}\ \parr}{\nu y.([\![M]\!]y \mid y(x).x().z \leftrightarrow y) \vdash [\![\Phi]\!]^\perp, z : [\![T]\!]}\ \mathsf{Cut}$$

# Type Preservation

**Theorem**

(Translation preserves types)

If $\Phi \vdash M : T$

then $[\![ M ]\!] z \vdash [\![ \Phi ]\!]^{\perp}, z : [\![ T ]\!]$.

# Part III

# Conclusions and future work

# Paradoxical combinator

$$X = X \supset A$$

$$\dfrac{\dfrac{\dfrac{[x : X \supset A]^x \quad [x : X]^x}{x\; x : A}\;\supset\text{-E}}{\lambda x.\, x\; x : X \supset A}\;\supset\text{-I}^x \qquad \dfrac{\dfrac{[x : X \supset A]^x \quad [x : X]^x}{x\; x : A}\;\supset\text{-E}}{\lambda x.\, x\; x : X}\;\supset\text{-I}^x}{(\lambda x.\, x\; x)\; (\lambda x.\, x\; x) : A}\;\supset\text{-E}$$

# Fixpoint combinator

$$X = X \supset A$$

$$\dfrac{\dfrac{\dfrac{[f : A \supset A]^f \quad \dfrac{[x : X \supset A]^x \quad [x : X]^x}{x\ x : A}\supset\text{-E}}{f\ (x\ x) : A}\supset\text{-E}}{\lambda x.\, f\ (x\ x) : X \supset A}\supset\text{-I}^x \quad \dfrac{\dfrac{[f : A \supset A]^f \quad \dfrac{[x : X \supset A]^x \quad [x : X]^x}{x\ x : A}\supset\text{-E}}{f\ (x\ x) : A}\supset\text{-E}}{\lambda x.\, f\ (x\ x) : X \supset A}\supset\text{-I}^x}{\dfrac{(\lambda x.\, f\ (x\ x))\ (\lambda x.\, f\ (x\ x)) : A}{\lambda f.\, (\lambda x.\, f\ (x\ x))\ (\lambda x.\, f\ (x\ x)) : (A \supset A) \supset A}\supset\text{-I}^f}\supset\text{-E}$$

# Restoring the full power of $\pi$-calculus

- Mix rule, Girard (1987)

$$\frac{P \vdash \Gamma \quad Q \vdash \Delta}{P \mid Q \vdash \Gamma, \Delta} \ \mathsf{Mix}$$

- Binary Cut rule, Abramsky, Gay, and Nagarajan (1996)

$$\frac{P \vdash \Gamma, x : A, y : B \quad Q \vdash \Delta, x : A^{\perp}, y : B^{\perp}}{\nu x : A, y : B.(P \mid Q) \vdash \Gamma, \Delta} \ \mathsf{BiCut}$$