The Amalgamation Property for Semilinear Commutative Idempotent Residuated Lattices

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February, 2018

Overview

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The Framework

Residuated Lattices

A residuated lattice is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, \mathrm{e} \rangle$ such that

- $\langle A, \wedge, \vee \rangle$ is a lattice,
- $\langle A, \cdot, \mathrm{e} \rangle$ is a monoid,
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- Semilinearity: $\lambda_z((x \vee y) \backslash x) \vee \rho_w((x \vee y) \backslash y) \approx \mathrm{e}$.

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A variety of RLs is semilinear if and only if it is generated by its totally ordered elements (chains).

First Properties of Our Algebras

Lemma

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1. $x \wedge y \leqslant xy \leqslant x \vee y$.

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Given an idempotent RL $\mathbf{A}$ *and two elements* $x, y \in A$*:*

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3. $xy \leqslant \mathrm{e} \implies xy = x \wedge y$.
4. *In the positive cone,* $xy = x \vee y$.

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Given an idempotent RL $\mathbf{A}$ *and two elements* $x, y \in A$:

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4. *In the positive cone,* $xy = x \vee y$.
5. *In the negative cone,* $xy = x \wedge y$.

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Lemma

The product on an idempotent chain $\mathbf{A}$ *is conservative (i.e.,* $xy \in \{x, y\}$*). Thus,* $\sqsubseteq$ *determines the product of* $\mathbf{A}$.

Counting Commutative Idempotent Chains

Corollary

If $\mathbf{A}$ *is a commutative idempotent chain, then* $\langle A, \cdot, \mathrm{e} \rangle$ *is an upper-bounded meet-semilattice, with induced order* $\sqsubseteq$*, which is total.*

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Theorem

There are 2^{n-2} *commutative idempotent chains of size* $n \geqslant 2$*.*

A Couple of Tools: Sugihara Monoids and Nuclei

Sugihara Monoids

In a pointed residuated lattice $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, \mathrm{e}, 0\rangle$, we define:

$$\sim x = x \backslash 0 \quad \text{and} \quad -x = 0/x.$$

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A Sugihara monoid is a distributive involutive commutative idempotent RL. It is odd if $\mathrm{e} = 0$.

The variety of Sugihara monoids is semilinear, as proven by Dunn.

Nuclei

A nucleus is a closure operator on a RL such that $\gamma(x)\gamma(y) \leqslant \gamma(xy)$.

If γ is a nucleus, then $\mathbf{A}_\gamma = \langle A_\gamma, \wedge, \vee_\gamma, \circ_\gamma, \backslash, /, \gamma(\mathrm{e})\rangle$ is a residuated lattice, where

$$x \vee_\gamma y = \gamma(x \vee y) \quad \text{and} \quad x \circ_\gamma y = \gamma(x \cdot y).$$

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Example

Fix $a \in A$ and define $\gamma_a(x) = (a/x)\backslash a$. This is always a closure operator on $\mathbf{A}$ and satisfies

$$y \cdot \gamma_a(x) \leqslant \gamma_a(yx).$$

Thus, if $\mathbf{A}$ is commutative, then γ_a is a nucleus.

The Representation Theorem

Lemma

1. *If* $\mathbf{A}$ *is a commutative idempotent chain, then* $\mathbf{A}_{\gamma_e}$ *is a subalgebra of* $\mathbf{A}$*. In fact,* $\mathbf{A}_{\gamma_e}$ *is a retract of* $\mathbf{A}$*.*

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2. *Every homomorphism* $f : \mathbf{A} \to \mathbf{B}$ *between commutative idempotent chains restricts to a homomorphism*

$$f{\restriction}_{A_{\gamma_{\mathrm{e}}}} : \mathbf{A}_{\gamma_{\mathrm{e}}} \to \mathbf{B}_{\gamma_{\mathrm{e}}}.$$

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Theorem (A)

Given a commutative idempotent chain $\mathbf{A}$*:*

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4. *For every* $x, y \in A$, *with* $x \in A_c$,

$$x \backslash y = \begin{cases} {\sim}c \vee x \vee y & \text{if } x \leqslant y, \\ {\sim}c \wedge y & \text{if } y < x. \end{cases}$$

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Given:

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we define $A = \bigcup_{c \in S} X_c$, and for $x \in X_c$ and $y \in X_{c'}$,

$$x \leqslant y \iff \begin{cases} c < c', & \text{or} \\ c = c' \text{ and } x \leqslant_c y. \end{cases}$$

$$xy = \begin{cases} x \wedge y & \text{if } c = c' \leqslant \mathrm{e}, \\ x \vee y & \text{if } \mathrm{e} < c = c', \\ x & \text{if } c \neq c' \text{ and } cc' = c, \\ y & \text{if } c \neq c' \text{ and } cc' = c', \end{cases} \qquad x \backslash y = \begin{cases} {\sim}c \vee x \vee y & \text{if } x \leqslant y, \\ {\sim}c \wedge y & \text{if } y < x. \end{cases}$$

Representation Theorem for Commutative Idempotent Chains

Theorem (B)

The previous construction produces a commutative idempotent chain $\mathbf{A}$*. Moreover,* $\mathbf{A}_{\gamma_{\mathrm{e}}} = \mathbf{S}$ *and for every* $c \in S$*,* $A_c = X_c$*.*

The Amalgamation Property

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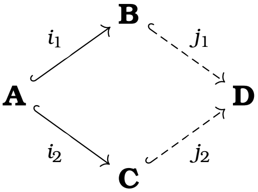
A class of algebras has the Amalgamation Property if every pair of embeddings of algebras of the class $\mathbf{A} \hookrightarrow \mathbf{B}$ and $\mathbf{A} \hookrightarrow \mathbf{C}$ admits an amalgam in the class, that is, an algebra $\mathbf{D}$ and two embeddings $\mathbf{B} \hookrightarrow \mathbf{D}$ and $\mathbf{C} \hookrightarrow \mathbf{D}$ rendering commutative the diagram:



Example

The class of totally ordered sets has the AP.

AP for Totally-Ordered Odd Sugihara Monoids

Lemma

Let $\langle A, \leqslant \rangle$ *be a total order and* ${\sim} : A \to A$ *and* $\mathrm{e} \in A$ *be s.t.:*

- $x \leqslant y \implies {\sim}y \leqslant {\sim}x$,
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Define $\cdot$ *and* $\backslash$ *on* A *as follows:*

$$x \cdot y = y \cdot x = \begin{cases} x \vee y & \text{if } \mathrm{e} \leqslant x, y \\ x \wedge y & \text{if } x, y \leqslant \mathrm{e} \\ x & \text{if } x \leqslant \mathrm{e} < y \text{ and } x \leqslant {\sim}y \\ y & \text{if } x \leqslant \mathrm{e} < y \text{ and } {\sim}y < x \end{cases}$$

$$x \backslash y = \begin{cases} {\sim}x \vee y & \text{if } x \leqslant y \\ {\sim}x \wedge y & \text{if } y \leqslant x. \end{cases}$$

Then, $\langle A, \wedge, \vee, \cdot, \backslash, \mathrm{e} \rangle$ *is a totally ordered odd Sugihara monoid.*

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Corollary

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Proof.

1. Forget the positive cones.
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3. Consider an amalgam $\mathbf{D}^-$ on $D^- = B^- \cup C^-$ as total ordered sets.
4. Using the involutions on $\mathbf{B}$ and $\mathbf{C}$, and the order of $\mathbf{D}$, define the corresponding total order on $D^+ = B^+ \cup C^+$, and take $\mathbf{D} = \mathbf{D}^- \oplus \mathbf{D}^+$.

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5. Define the multiplicative structure and the residual, by using the involution and the negative cone. □

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1. Consider the Sugihara monoid retracts $\mathbf{A}_{\gamma_e} \hookrightarrow \mathbf{B}_{\gamma_e}$ and $\mathbf{A}_{\gamma_e} \hookrightarrow \mathbf{C}_{\gamma_e}$.
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   - 3.3 $c \in A_{\gamma_e}$: define $\langle X_c, \leqslant_c \rangle$ as the amalgam of $\langle A_c, \leqslant \rangle \hookrightarrow \langle B_c, \leqslant \rangle$ and $\langle A_c, \leqslant \rangle \hookrightarrow \langle C_c, \leqslant \rangle$.

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Proof.

1. Consider the Sugihara monoid retracts $\mathbf{A}_{\gamma_e} \hookrightarrow \mathbf{B}_{\gamma_e}$ and $\mathbf{A}_{\gamma_e} \hookrightarrow \mathbf{C}_{\gamma_e}$.
2. Consider the amalgam of these embeddings $\mathbf{S}$.
3. For every $c \in S$, we have three possibilities:
   - 3.1 $c \in B_{\gamma_e} \setminus C_{\gamma_e}$: define $\langle X_c, \leqslant_c \rangle = \langle B_c, \leqslant \rangle$;
   - 3.2 $c \in C_{\gamma_e} \setminus B_{\gamma_e}$: define $\langle X_c, \leqslant_c \rangle = \langle C_c, \leqslant \rangle$;
   - 3.3 $c \in A_{\gamma_e}$: define $\langle X_c, \leqslant_c \rangle$ as the amalgam of $\langle A_c, \leqslant \rangle \hookrightarrow \langle B_c, \leqslant \rangle$ and $\langle A_c, \leqslant \rangle \hookrightarrow \langle C_c, \leqslant \rangle$.
4. Using $\mathbf{S}$ and $\{\langle X_c : c \in S \rangle\}$, construct a commutative idempotent chain. □

AP for Semilinear Commut. Idempotent RLs

Theorem (Metcalfe, Montagna, Tsinakis)

1. *Let* V *be a variety of semilinear RLs with the congruence extension property*
2. *let* $\mathcal{T}$ *be the class of all fin. generated chains of* V.

If every span in $\mathcal{T}$ *has an amalgam in* V*, then* V *has the amalgamation property.*

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Corollary

The variety of semilinear commutative idempotent residuated lattices has the AP.

Future Work: The Noncommutative Case

Thank you for your attention!