Structural rules for multi-valued logics

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Reminder

– Usually, the term *structural rules* refers to the following rules, devised by Gentzen for incorporation in his sequent calculi LJ/LK for intuitionistic/classical logics.

$$\frac{\Gamma_1, \psi, \varphi, \Gamma_2 \rightarrow \chi}{\Gamma_1, \varphi, \psi, \Gamma_2 \rightarrow \chi}(PL) \quad \frac{\Gamma \rightarrow \chi}{\Gamma, \varphi \rightarrow \chi}(WL) \quad \frac{\Gamma, \varphi, \varphi \rightarrow \chi}{\Gamma, \varphi \rightarrow \chi}(CL) \tag{1}$$

- Here Γ is a meta-variable over sequences of object language formulas, and φ, ψ and χ are meta-variables ranging over formulas.
- In the case of *multi-conclusions* sequents $\Gamma \rightarrow \Delta$, there are also analogous rules (PR), (WR), (CR) for modifying Δ, the r.h.s. of a sequent.
– Initial sequents $\Gamma, \varphi \rightarrow \varphi$ or $\Gamma, \varphi \rightarrow \varphi, \Delta$.

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$$\frac{\Gamma \rightarrow \varphi \quad \Gamma, \varphi \rightarrow \psi}{\Gamma \rightarrow \psi}(cut) \quad (one\ version)$$

– The characteristic property of those structural rules, inspiring their names, is that they do not refer to any *specific* connective/quantifier, in contrast to logical (or operational) rules, defining the meanings of the logical operators. However, their soundness does depend on the logics for which they are intended to be *bivalent*.

Our Aim

– In this talk, we consider structural rules suitable for *multi-valued logics*, rules endowed with the same characteristic of not depending on the operators of the logic.
- 1) Adapting Gentzen's rules to multi-valued logics.

- 2) Adding new rules, suitable *only* to multi-valued logics.

– Since the semantics is multi-valued, some syntactic means is needed for referring to the truth value of a formula, in particular within a derivation a proof-system.

– We propose such a generalization of a traditional sequent, and consider suitable structural rules applicable to this generalized structure.

Located formulas and sequents

– Let $\mathcal{V} = \{v_1, \cdots, v_n\},\ n \geq 2$ be the (unordered) collection of truth-values underlying a multi-valued propositional logic L, given by *truth-tables* for the connectives. Let $\widehat{n} = \{1, \cdots, n\}$.
– A *located formula (l-formula)* is a pair (φ, k), where φ is a formula and $k \in \widehat{n}$.
- The intended meaning of (φ, k) is the association of φ with the truth-value $v_k \in \mathcal{V}$.
– A *located sequent (l-sequent)* has the form $\Pi = \Gamma \rightarrow \Delta$, where $\Gamma,\ \Delta$ are (possibly empty) finite sets of l-formulas. We use $\mathbf{\Pi}$ for sets of l-sequents.
– Let σ range over truth-value assignments, mapping formulas to truth-values; for atomic sentences the mapping is arbitrary, and is extended to formulas so as to respect the truth-tables of the operators.

Definition (satisfaction, consequence):

$$\sigma \models \Pi \text{ iff } \begin{array}{l} \text{if } \sigma[\![\varphi]\!] = v_k \text{ for all } (\varphi, k) \in \Gamma, \\ \text{then } \sigma[\![\psi]\!] = v_j \text{ for some } (\psi, j) \in \Delta \end{array} \quad (2)$$

$$\mathbf{\Pi} \models \Pi \text{ iff } \sigma \models \Pi' \text{ for all } \Pi' \in \mathbf{\Pi} \text{ implies } \sigma \models \Pi \quad (3)$$

Adapting Gentzen's rules

– We present elsewhere ND proof-systems sound and complete for this consequence relation. Logical rules for arbitrary logical operators are constructed from the truth-tables in a uniform way.

– The structural rules below are all sound for the multi-valued semantics. As we are interested here in structural rules only, we skip the discussion of logical rules.

– **Adapted Gentzen rules:**

$$\frac{\Gamma \to (\chi, j)}{\Gamma, (\varphi, i) \to (\chi, j)}\,(WL_i) \quad \frac{\Gamma, (\varphi, i), (\varphi, i) \to (\chi, j)}{\Gamma, (\varphi, i) \to (\chi, j)}\,(CL_i) \tag{4}$$

– The *initial l-sequents* are the natural generalization of Gentzen's bivalent identity rule

$$(\varphi, k) \to (\varphi, k), \text{ for each } k \in \widehat{n} \tag{5}$$

or

$$\Gamma, (\varphi, k) \to (\varphi, k), \Delta \text{ for each } k \in \widehat{n} \tag{6}$$

Shifting: a *new* family of rules for multi-valuedness

– For every $1 \leq i, j \leq n$:

$$\frac{\Gamma, (\varphi, i) \to \Delta}{\Gamma \to \Delta, \{\varphi\} \times \overline{\{i\}}} (\overrightarrow{s}_i) \qquad \frac{\Gamma \to \Delta, (\varphi, i)}{\Gamma, (\varphi, j) \to \Delta} (\overleftarrow{s}_{i,j}), j \neq i \tag{7}$$

– To understand these rules, consider again Gentzen's rules for negation:

$$\frac{\Gamma \to \varphi, \Delta}{\Gamma, \neg\varphi \to \Delta} (\neg L) \qquad \frac{\Gamma, \varphi \to \Delta}{\Gamma \to \neg\varphi, \Delta} (\neg R) \tag{8}$$

The effect of these rules in classical, bivalent logic is that φ is false iff $\neg\varphi$ is true.

– Suppose we want to avoid the use of negation, and, instead, refer directly to the falsity of a formula. Then, a reformulation of Gentzen's rule using l-sequents with $n = 2$, and using $t, \ f$ as mnemonic locating indices (instead of $1, \ 2$), is as follows.

$$\frac{\Gamma \to (\varphi, t), \Delta}{\Gamma, (\varphi, f) \to \Delta} (\overleftarrow{s}_{t,f}) \qquad \frac{\Gamma, (\varphi, t) \to \Delta}{\Gamma \to (\varphi, f), \Delta} (\overrightarrow{s}_t)$$

$$\frac{\Gamma \to (\varphi, f), \Delta}{\Gamma, (\varphi, t) \to \Delta} (\overleftarrow{s}_{f,t}) \qquad \frac{\Gamma, (\varphi, f) \to \Delta}{\Gamma \to (\varphi, t), \Delta} (\overrightarrow{s}_f) \tag{9}$$

Shifting - continued

– In this case, a logical rule (about negation) is converted into structural rules (about truth and falsity).
- Here there is *only one way to shift*. Note that each negation rule breaks into two separate shifting rules, as truth and falsity need to be treated separately.
- The shifting rules above consider the general case, where not having some given truth-value (out of n values) is related to having other truth values.

An l-sequent immediately derivable by shifting is $\rightarrow\{\varphi\} \times \widehat{n}$ (where $\widehat{n} = \{1, \cdots, n\}$):

$$\frac{(\varphi, k)\rightarrow(\varphi, k)}{\rightarrow\{\varphi\} \times \widehat{n}} \, (\overrightarrow{s}_k) \qquad (10)$$

(for an arbitrary $k \in \widehat{n}$).
- This is a natural generalization of the *excluded middle*, the latter seen as stating that every φ is either true or false. Here, every φ has *some* truth value within $\mathcal{V}$.

Coordination rules

– The other central (family of) structural rules are the *coordination rules*:

$$\frac{\Gamma_1 \rightarrow (\varphi, i), \Delta_1 \quad \Gamma_2 \rightarrow (\varphi, j), \Delta_2}{\Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2} (c_{i,j}), \; \{i, j\} \subseteq \hat{n}, \; i \neq j \tag{11}$$

– These rules are a natural generalization of *ex contradictione (sequitur) quodlibet*, everything follows from a contradiction.

- A contradiction can be regarded as some φ being both true and false (and no need for negation!).

- Its generalization is some φ having two *different* truth values $v_i, \; v_j \in \mathcal{V}, \; i \neq j$.

Conclusion

– There remains an issue of sub-structurality regarding those multi-valued structural rules:

- The coordination rules ($c_{i,j}$) were formulated as *multiplicative* (context independent) rules. When is an additive formulation (context sharing) inequivalent?

- What happens when either the switching rules or the coordination rules (or both) are *omitted*?