An Abstract Approach to Consequence Relations II

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Tarskian consequence

A Tarskian consequence relation (TCR) on $\mathcal{L}$-formulas is a relation $\vdash \subseteq \wp(Fm_{\mathcal{L}}) \times Fm_{\mathcal{L}}$ such that for all $\Gamma \cup \Delta \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{L}}$:

1. $\Gamma \vdash \varphi$ whenever $\varphi \in \Gamma$ (Reflexivity)
2. If $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \varphi$ (Monotonicity)
3. If $\Delta \vdash \psi$ and $\Gamma \vdash \varphi$ for every $\varphi \in \Delta$, then $\Gamma \vdash \psi$ (Cut)

A TCR is substitution-invariant if $\Gamma \vdash \varphi$ implies $\sigma(\Gamma) \vdash \sigma(\varphi)$ for all $\mathcal{L}$-substitutions σ ($\sigma(\Gamma)$ defined pointwise).

Blok–Jónsson consequence: The vanilla theory

An abstract consequence relation (ACR) over the set X is a relation $\vdash \subseteq \wp(X) \times X$ such that for all $\Gamma \cup \Delta \cup \{a\} \subseteq X$:

1. $\Gamma \vdash a$ whenever $a \in \Gamma$ (Reflexivity)
2. If $\Gamma \vdash a$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash a$ (Monotonicity)
3. If $\Delta \vdash a$ and $\Gamma \vdash b$ for every $b \in \Delta$, then $\Gamma \vdash a$ (Cut)

Similarity of acr's

Acr's $\vdash_1$ and $\vdash_2$ over X_1 and X_2 resp. are similar if there are mappings

$$\tau\colon X_1 \to \wp(X_2) \qquad\qquad \rho\colon X_2 \to \wp(X_1)$$

such that for every $\Gamma \cup \{a\} \subseteq X_1$ and every $\Delta \cup \{b\} \subseteq X_2$:

S1	$\Gamma \vdash_1 a$ iff $\tau(\Gamma) \vdash_2 \tau(a)$	S3	$a \dashv\vdash_1 \rho(\tau(a))$
S2	$\Delta \vdash_2 b$ iff $\rho(\Delta) \vdash_1 \rho(b)$	S4	$b \dashv\vdash_2 \tau(\rho(b))$

Put differently, the ACR's $\vdash_1$ and $\vdash_2$ are similar when:

- $\vdash_1$ is faithfully translatable via the mapping τ into $\vdash_2$ (S1)
- $\vdash_2$ is faithfully translatable via the mapping ρ into $\vdash_1$ (S2)
- the two mappings ρ and τ are mutually inverse (S3 and S4)

Examples of similarities

- Algebraisability (similarity between a TCR and the equational consequence relation of some class of algebras);
- Gentzenisability (similarity between a TCR and some consequence relations on sequents);
- Same-environment similarities (e.g. algebraisable TCR's that have the same equivalent algebraic semantics with different transformers).

Limits of the vanilla theory

The set X is a "black box": it carries no inner structure, whence e.g. we can give no notion of endomorphism other than the trivial one (a permutation). Substitution-invariance cannot simply be expressed.

With respect to their Tarskian competitor, Blok and Jónsson have attained a greater level of generality at the expense of the applicability of the theory (Hilbert systems, matrices, etc.)

Action-invariant ACR's

The monoid $\mathbf{M} = (M, \circ, 1)$ is said to act on non-empty set X if there is an operation $\cdot : M \times X \to X$ such that, for all $\sigma, \sigma' \in M$ and all $a \in X$:

$$(\sigma \circ \sigma') \cdot a = \sigma \cdot (\sigma' \cdot a) .$$

The operation $\cdot$ is called scalar product, and the scalars in M are called actions. We write $\sigma(a)$ instead of $\sigma \cdot a$.

When $\mathbf{M}$ acts on X, an ACR $\vdash$ on X is called action-invariant if, for any $\sigma \in M$, for any $\Gamma \subseteq X$ and for any $a \in X$,

$$\text{if } \Gamma \vdash a, \text{ then } \sigma(\Gamma) \vdash \sigma(a) .$$

The general theory (BJ, Galatos–Tsinakis)

- Consider symmetric (multiple-conclusion) versions of the ACR's;
- "Lift" the actions and the transformers to the level of powersets;
- $\wp(M)$ is the universe a complete residuated lattice, with complex product as the residuated operation (the scalars); $\wp(X)$ is the universe of a complete lattice (the vectors); Scalar product is a biresiduated map that satisfies the usual properties of a monoid action.
- Go fully abstract: ACR's on complete lattices as preorders on complete lattices that contain the converse of the lattice order.
- Abstractly, equivalence of such ACR's can be defined by tweaking similarity in such a way as to accommodate action-invariance.

Limits of the general theory

The idea of a consequence relation as a preorder on a complete lattice that contains the converse of the lattice order is not general enough: it rules out important cases where we have non-idempotent operations of premiss and conclusion aggregation.

Example: multiset consequence (internal consequence relations of substructural sequent calculi, resource-conscious versions of logics from commutative integral residuated lattices, etc.) can be only treated as consequence relation on sequents but not as consequence relation on formulas

So we could use the theory of algebraization of Gentzen systems but this would add an unnecessary level of complexity . . .

Deductive relations

Definition

A deductive relation (DR) $\vdash$ on a dually integral Abelian po-monoid $\mathbf{R} = \langle R, \leq, +, 0\rangle$ is a preorder on R such that for every $a, b, c \in R$:

1. If $a \leq b$, then $b \vdash a$.
2. If $a \vdash b$, then $a + c \vdash b + c$.

Examples (1)

Example (Tarski)

Any TCR $\vdash$ on the language $\mathcal{L}$ canonically gives rise to a DR on the Abelian po-monoid

$$\mathbf{R} = \langle \wp\left(Fm_{\mathcal{L}}\right), \subseteq, \cup, \emptyset \rangle .$$

Example (Blok–Jónsson)

Any ACR $\vdash$ over the set X canonically gives rise to a DR on the Abelian po-monoid

$$\mathbf{R} = \langle \wp\left(X\right), \subseteq, \cup, \emptyset \rangle .$$

Examples (2)

Example (Multiset consequence)

Let $\mathcal{L}$ be a language, and let $Fm^{\flat}_{\mathcal{L}}$ be the set of finite multisets of $\mathcal{L}$-formulas. A multiset deductive relation (MDR) on $\mathcal{L}$ is a preorder $\vdash$ on $Fm^{\flat}_{\mathcal{L}}$ that satisfies the following additional postulates:

1. If $[\varphi_1, \ldots, \varphi_n] \leq [\psi_1, \ldots, \psi_m]$, then $[\psi_1, \ldots, \psi_m] \vdash [\varphi_1, \ldots, \varphi_n]$.
2. If $[\psi_1, \ldots, \psi_m] \vdash [\varphi_1, \ldots, \varphi_n]$, then

$$[\gamma_1, \ldots, \gamma_m] \uplus [\psi_1, \ldots, \psi_m] \vdash [\gamma_1, \ldots, \gamma_m] \uplus [\varphi_1, \ldots, \varphi_n].$$

So, any MDR $\vdash$ on the language $\mathcal{L}$ is a DR on

$$\mathbf{R} = \left\langle Fm^{\flat}_{\mathcal{L}}, \leq, \uplus, \emptyset \right\rangle.$$

$(\mathfrak{X} \uplus \mathfrak{Y}(\varphi) = \mathfrak{X}(\varphi) + \mathfrak{Y}(\varphi)$; $\mathfrak{X} \leq \mathfrak{Y}$ iff for all φ, $\mathfrak{X}(\varphi) \leq \mathfrak{Y}(\varphi))$.

Deductive operators

Definition

A deductive operator on a dually integral Abelian po-monoid $\mathbf{R} = \langle R, \leq, +, 0 \rangle$ is a map $\delta \colon R \to \mathcal{P}(R)$ such that for every $a, b, c \in R$:

1. $a \in \delta(a)$.
2. If $a \leq b$, then $\delta(a) \subseteq \delta(b)$.
3. If $a \in \delta(b)$, then $\delta(a) \subseteq \delta(b)$.
4. If $a \in \delta(b)$, then $a + c \in \delta(b + c)$.

Theorem

If $\mathbf{R} = \langle R, \leq, +, 0 \rangle$ is a dually integral Abelian po-monoid, then the lattices $\langle \mathrm{Rel}(\mathbf{R}), \subseteq \rangle$ and $\langle \mathrm{Oper}(\mathbf{R}), \preccurlyeq \rangle$ are isomorphic.

Partially ordered semirings

Definition

A partially ordered semiring (po-semiring) is a structure $\mathbf{A} = \langle A, \leq, +, \cdot, 0, 1 \rangle$ where:

1. $\langle A, \cdot, 1 \rangle$ is a monoid;
2. $\langle A, \leq, +, 0 \rangle$ is an Abelian po-monoid;
3. $\sigma \cdot 0 = 0 \cdot \sigma = 0$ for all $\sigma \in A$;
4. for every $\sigma, \pi, \varepsilon \in A$ we have
$$\pi \cdot (\sigma + \varepsilon) = (\pi \cdot \sigma) + (\pi \cdot \varepsilon) \text{ and } (\sigma + \varepsilon) \cdot \pi = (\sigma \cdot \pi) + (\varepsilon \cdot \pi).$$
5. if $\sigma \leq \pi$ and $0 \leq \varepsilon$, then $\sigma \cdot \varepsilon \leq \pi \cdot \varepsilon$ and $\varepsilon \cdot \sigma \leq \varepsilon \cdot \pi$.

A po-semiring $\mathbf{A} = \langle A, \leq, +, \cdot, 0, 1 \rangle$ is dually integral iff $\langle A, \leq, +, 0 \rangle$ is dually integral as a po-monoid.

Po-semirings of substitutions

Example

Let $\mathrm{Subst}(\mathbf{Fm}_{\mathcal{L}})$ be the set of *substitutions* of $\mathbf{Fm}_{\mathcal{L}}$. The structure

$$\mathbf{\Sigma} = \langle \mathrm{Subst}(\mathbf{Fm}_{\mathcal{L}})^{\flat}, \leq, \uplus, \cdot, 0, 1 \rangle,$$

where, for $\mathfrak{X} = [\sigma_1, \ldots, \sigma_n]$, $\mathfrak{Y} = [\pi_1, \ldots, \pi_m]$, $\sigma \in \mathrm{Subst}(Fm_{\mathcal{L}})$,

$$\mathfrak{X} \cdot \mathfrak{Y} = [\sigma_1 \circ \pi_1, \ldots, \sigma_1 \circ \pi_m, \ldots, \sigma_n \circ \pi_1, \ldots, \sigma_n \circ \pi_m],$$

$$1(\sigma) = \begin{cases} 1, \text{ if } \sigma = id_{Fm_{\mathcal{L}}} \\ 0, \text{ otherwise,} \end{cases}$$

$$0(\sigma) = 0,$$

is a dually integral po-semiring.

Modules over po-semirings

Definition

Let **A** be a dually integral po-semiring. An **A**-*module* is a structure $\boldsymbol{R} = \langle R, \leq, +, 0, * \rangle$ where $\langle R, \leq, +, 0 \rangle$ is a dually integral Abelian po-monoid and $*\colon A \times R \to R$ is a map that is order-preserving in both coordinates, and s.t.

1. $(\sigma \cdot \pi) * a = \sigma * (\pi * a)$;
2. $1 * a = a$;
3. $0^{\mathbf{A}} * a = 0^{\mathbf{R}}$;
4. $(\sigma * a) +^{\boldsymbol{R}} (\sigma * b) = \sigma * (a +^{\boldsymbol{R}} b)$;
5. $(\sigma +^{\mathbf{A}} \pi) * a = (\sigma * a) +^{\boldsymbol{R}} (\pi * a)$.

Modules of substitutions

Example

Consider

$$\mathbf{\Sigma} = \langle \mathrm{Subst}(\mathbf{Fm}_{\mathcal{L}})^{\flat}, \leq, \uplus, \cdot, 0, 1 \rangle,$$

and let $\boldsymbol{R} = \left\langle Fm^{\flat}_{\mathcal{L}}, \leq, \uplus, \emptyset, * \right\rangle$, where for

$$\sigma = [\sigma_1, \ldots, \sigma_n] \text{ and } \varphi = [\varphi_1, \ldots, \varphi_m].$$

we set

$$\sigma * \varphi = [\sigma_1(\varphi_1), \ldots, \sigma_1(\varphi_m), \ldots, \sigma_n(\varphi_1), \ldots, \sigma_n(\varphi_m)].$$

$\boldsymbol{R}$ is a $\mathbf{\Sigma}$-module.

Action-invariant deductive operators on modules over po-semirings

Definition

An action-invariant deductive operator on an **A**-module $\boldsymbol{R} = \langle R, \leq, +, 0, * \rangle$ is a deductive operator δ on $\langle R, \leq, +, 0 \rangle$ such that for every $\sigma \in A$ and $a, b \in R$:

$$\text{if } a \in \delta(b), \text{ then } \sigma * a \in \delta(\sigma * b).$$

The category of A-modules

A-Md is the category whose objects are **A**-modules and whose arrows are po-monoid homomorphisms τ that respect the monoidal action:

$$\tau(\sigma * a) = \sigma * \tau(a) \text{ for every } \sigma \in A \text{ and } a \in R.$$

Theories

Lemma

Let δ be an action-invariant deductive operator on the $\mathbf{A}$-module $\boldsymbol{R}$. The structure

$$\boldsymbol{R}_\delta = \langle \delta[R], \subseteq, +^\delta, \delta(0), *^\delta \rangle$$

where $\delta(a) +^\delta \delta(b) = \delta(a+b)$ and $\sigma *^\delta \delta(a) = \delta(\sigma * a)$, is an object of $\mathbf{A}$-Md and the map $\delta \colon \boldsymbol{R} \to \boldsymbol{R}_\delta$ is an arrow of $\mathbf{A}$-Md.

Structural representations

Definition

Let δ and γ be two action-invariant deductive operators on the $\mathbf{A}$-modules $\boldsymbol{R}$ and $\boldsymbol{S}$, respectively. A structural representation of δ into γ is an injective morphism $\Phi\colon \boldsymbol{R}_\delta \to \boldsymbol{S}_\gamma$ that reflects the order.

Definition

An $\mathbf{A}$-module $\boldsymbol{R}$ has the representation property if for any $\mathbf{A}$-module $\boldsymbol{S}$ and action-invariant deductive operators δ and γ on $\boldsymbol{R}$ and $\boldsymbol{S}$ respectively, every structural representation of δ into γ is induced, i.e. there is a morphism $\tau\colon \boldsymbol{R} \to \boldsymbol{S}$ that makes the following diagram commute:

$$\begin{array}{ccc} \boldsymbol{R} & \xrightarrow{\ \tau\ } & \boldsymbol{S} \\ \big\downarrow{\scriptstyle \delta} & & \big\downarrow{\scriptstyle \gamma} \\ \boldsymbol{R}_\delta & \xrightarrow{\ \Phi\ } & \boldsymbol{S}_\gamma \end{array}$$

Motivating example

Theorem

The $\mathbf{\Sigma}$*-module*

$$\boldsymbol{R} = \left\langle Fm^{\flat}_{\mathcal{L}}, \uplus, \emptyset, \leq, * \right\rangle$$

of finite multisets of formulas of a sentential language has the representation property.

Axiomatic system

Let us fix a propositional language

A consecution: a pair of finite multisets of formulas, written as $\Gamma \rhd \Delta$

A consecution is single-conclusion if $\Delta = [\varphi]$

We identify the consecution $\emptyset \rhd [\varphi]$ with the formula φ.

(Single-conclusion) axiomatic system: a set of (single-conclusion) consecutions closed under arbitrary substitutions

Two notions of proof — tree proof

Definition (Tree-proof)

Let AS be a single-conclusion axiomatic system in $\mathcal{L}$. A tree-proof of a formula φ from a multiset of formulas Γ in AS is a finite tree t labelled by formulas such that:

- The root of t is labelled by φ.
- If a leaf of t is labelled by ψ, then either
  - ψ is an axiom or
  - ψ is an element of Γ and it labels at most $\Gamma(\psi)$ leaves in t.
- If a node of t is labelled by ψ and $\Delta \neq \emptyset$ is the multiset of labels of its predecessor nodes, then $\Delta \rhd [\psi] \in$ AS.

We write $\Gamma \vdash^{t}_{\mathrm{AS}} \varphi$ whenever there is a tree-proof of φ from Γ in AS.

Two notions of proof — derivation

Definition (Derivation)

Let AS be an axiomatic system in $\mathcal{L}$. A derivation of a finite multiset of formulas Δ from a finite multiset of formulas Γ in AS is a finite sequence $\langle \Gamma_1, \ldots, \Gamma_n \rangle$ of finite multisets of formulas such that:

- $\Gamma_1 = \Gamma$;
- For every Γ_j, $1 < j \leq n$, there is $\Psi \rhd \Psi' \in \mathrm{AS}$, such that $\Psi \leqslant \Gamma_{j-1}$ and $\Gamma_j = (\Gamma_{j-i} \setminus \Psi) \uplus \Psi'$;
- $\Delta \leqslant \Gamma_n$.

We write $\Gamma \vdash_{\mathrm{AS}} \Delta$, if there is a derivation of Δ from Γ in AS.

Some observations

Lemma

Let The relation $\vdash_{\mathsf{AS}}$ *is the least substitution-invariant* MDR *containing an axiomatic system* AS*.*

Definition

Axiomatic system AS is a presentation of an substitution-invariant MDR $\vdash$ if $\vdash = \vdash_{\mathsf{AS}}$.

Clearly, due to the previous lemma, each MDR can be seen as its own presentation, and so we obtain:

Corollary (Łos–Suszko)

Every substitution-invariant MDR $\vdash$ *coincides with the derivability relation* $\vdash_{\mathsf{AS}}$ *of some axiomatic system* AS*.*

Relationship between derivations and tree-proofs

Lemma

Let AS *be a single-conclusion axiomatic system,* $\Gamma \vdash_{\mathsf{AS}} \Delta$*, and* $\varphi \in |\Delta|$*. Then there are multisets of* $\mathcal{L}$*-formulas* Γ^{φ} *and* Γ^{r} *such that* $\Gamma^{\varphi} \uplus \Gamma^{r} = \Gamma$*,* $\Gamma^{\varphi} \vdash^{t}_{\mathsf{AS}} \varphi$ *and* $\Gamma^{r} \vdash_{\mathsf{AS}} \Delta \setminus [\varphi]$*.*

Corollary

Let AS *be a single-conclusion axiomatic system. Then* $\vdash^{t}_{\mathsf{AS}}$ *is a single-conclusion* MDR *and* $\vdash_{\mathsf{AS}}$ *is the least* MDR $\vdash$ *such that*

$$\Gamma \vdash^{t}_{\mathsf{AS}} \alpha \text{ iff } \Gamma \vdash [\alpha] \, .$$

Recall that $[p \otimes q] \vdash_{\mathcal{MV}} [p, q]$, and neither $\nvdash_{\mathcal{MV}} [p]$ nor $\nvdash_{\mathcal{MV}} [q]$ thus:

Corollary

There is no single-conclusion presentation of $\vdash_{\mathcal{MV}}$*.*

Relationship between derivations and tree-proofs

Definition

The single conclusion axiomatic system MV^s contains as axioms all instances of the axioms of Łukasiewicz logic and the sole deduction rule:

$$[\varphi, \varphi \to \psi] \rhd [\psi].$$

The axiomatic system MV is an extension of MV^s by the rule:

$$[\varphi \otimes \psi] \rhd [\varphi, \psi].$$

Theorem

Let $\Gamma, \Delta, [\varphi] \in Fm^{\flat}_{\mathcal{L}_0}$. *Then*

- $\Gamma \vdash_{\mathsf{MV}^s} [\varphi]$ *iff* $\Gamma \vdash_{\mathcal{MV}} [\varphi]$.
- $\Gamma \vdash_{\mathsf{MV}} \Delta$ *iff* $\Gamma \vdash_{\mathcal{MV}} \Delta$.

A little bit of propaganda I

Algebra and substructural logics

(Take 6)

Cagliari, June 11–13

A little bit of propaganda II

Logic, Algebra, Truth Degrees
&
Advances in Modal Logic

Bern, August 27–31