Duality for Płonka sums

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(Partly joint work with A. Loi and L. Peruzzi)

SYSMICS Workshop 2018

Outline

1. Płonka sums (and their logical application)
2. Semilattices inverse/direct systems.
3. Duality.

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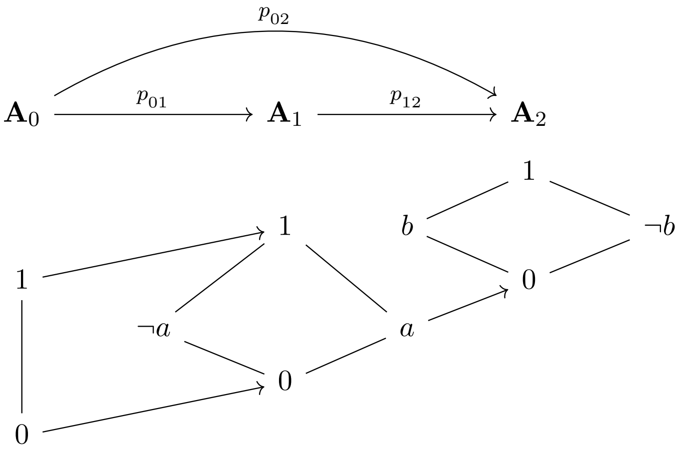
Płonka sums: example



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I is called the *index set* of the system $\mathcal{X}$, X_i are the *terms* and $p_{ii'}$ are referred to as *bonding morphisms* of $\mathcal{X}$.

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A *morphism* between two *strongly inverse systems* $\mathcal{X} = \langle X_i, p_{ii'}, I \rangle$ and $\mathcal{Y} = \langle Y_j, q_{jj'}, J \rangle$, is a pair (φ, f_j) s.t.

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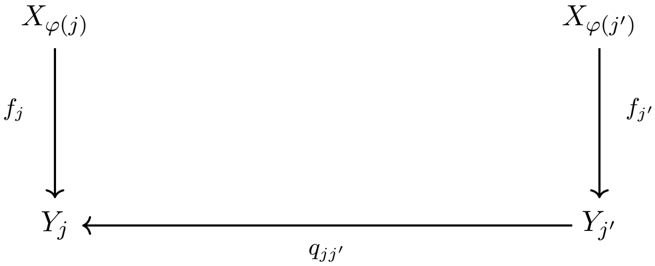

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$$\begin{array}{ccc} X_{\varphi(j)} & \xleftarrow{\;p_{\varphi(j)\varphi(j')}\;} & X_{\varphi(j')} \\ \Big\downarrow{\scriptstyle f_j} & & \Big\downarrow{\scriptstyle f_{j'}} \\ Y_j & \xleftarrow[\;q_{jj'}\;]{} & Y_{j'} \end{array}$$

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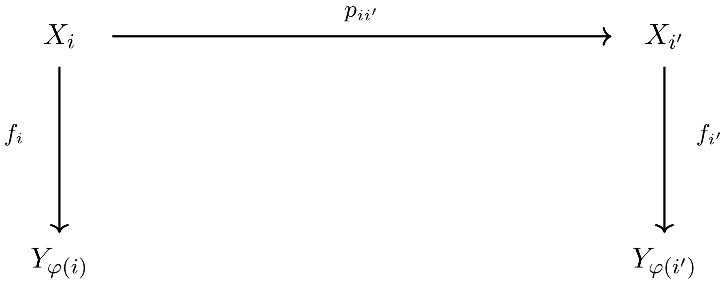

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Duality

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A category $\mathfrak{D}$ is the *dual category* of $\mathfrak{C}$, if there exists an invertible contravariant functor $\mathcal{F} : \mathfrak{C} \to \mathfrak{D}$ with inverse $\mathcal{G}$ s.t. $\mathcal{G} \circ \mathcal{F} = id_{\mathfrak{C}}$ and $\mathcal{G} \circ \mathcal{F} = id_{\mathfrak{D}}$.

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Remark

If $\mathfrak{C}$ and $\mathfrak{D}$ are dual categories, then strong-dir-$\mathfrak{C}$ is the dual category of strong-inv-$\mathfrak{D}$.

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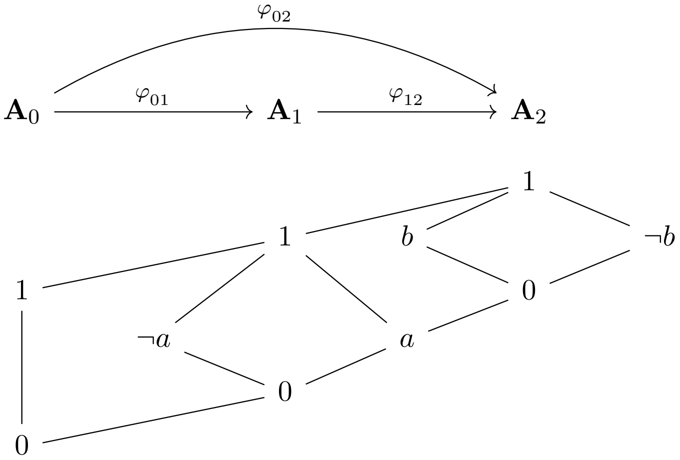
In case ν contains constants, then we define $g = g^{\mathbf{A}_0}$.

Example



φ_{02}
$\mathbf{A}_0$
φ_{01}
$\mathbf{A}_1$
φ_{12}
$\mathbf{A}_2$
1
1
b
$\neg b$
1
0
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a
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Płonka sums representation

Theorem

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2. *If $\mathbf{B}$ is an involutive bisemilattice, then $\mathbf{B}$ is isomorphic to the Płonka sum over a strongly direct system of Boolean algebras.*

Categories into play

Category	Objects	Morphisms
$\mathfrak{BA}$	Boolean Algebras	Homomorph. of $\mathcal{BA}$
$\mathfrak{IBSL}$	Involutive bisemilattices	Hom. of $\mathcal{IBSL}$
strong-dir-$\mathfrak{BA}$	str. dir. systems of B.A.	Morphisms of s.d.s.
$\mathfrak{SA}$	Stone spaces	continuous maps
strong-inv-$\mathfrak{SA}$	str. inv. systems of Stone sp.	Morphisms of s.i.s.

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Is it possible to describe the dual in terms of a unique space?

Duality for $\mathcal{BSL}$

Theorem (Gierz, Romanowska)

The categories $\mathfrak{DB}$ *and* $\mathfrak{GR}$ *are dual to each other under the functors* $\mathrm{Hom_b}(-, \mathbf{3}) : \mathfrak{DB} \to \mathfrak{GR}$ *and* $\mathrm{Hom_{GR}}(-, \mathbf{3}) : \mathfrak{GR} \to \mathfrak{DB}$.

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2. $\neg(a * b) = \neg a * \neg b$
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6. there exist $\varphi_0, \varphi_1 \in \mathrm{Hom}_{\mathrm{GR}}(\mathbf{G}, \mathbf{3})$ s.t. $\neg\varphi_0 = \varphi_1$ and $\varphi + \varphi_0 = \varphi$, for each $\varphi \in \mathrm{Hom}_{\mathrm{GR}}(\mathbf{G}, \mathbf{3})$.

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Corollary

The category strong-inv-$\mathfrak{SA}$ is equivalent to the category $\mathfrak{IGR}$.

Thank you!