The classification of all the subvarieties of $\mathbb{DNMG}$

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joint work with

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Weak Nilpotent Minimum logic WNM was introduced in [EG01], as a weakening of NM (also introduced in [EG01]), the logic of Nilpotent Minimum t-norm [Fod95].

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During the years a lot of research has been done, by classifying many axiomatic extensions of WNM, as well as subvarieties of $\mathbb{WNM}$. Among them, we mention the varieties of Gödel algebras $\mathbb{G}$, Nilpotent Minimum algebras $\mathbb{NM}$, and of Drastic Product algebras $\mathbb{DP}$.

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In this talk we will present the variety $\mathbb{DNMG}$, one of the smallest within $\mathbb{WNM}$ which contains $\mathbb{G}$, $\mathbb{NM}$, $\mathbb{DP}$, and it is generated by a single chain.

We will classify and axiomatize all the $\mathbb{DNMG}$ subvarieties, by showing that they are countably-many, and we will describe the structure of the lattice of the subvarieties of $\mathbb{DNMG}$.

A WNM-algebra is an MTL-algebra $(A, *, \to, \wedge, \vee, 0, 1)$ satisfying

(WNM) $$\neg(x * y) \vee ((x \wedge y) \to (x * y)) = 1.$$

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The operations $*, \to$ of a WNM-chain $\mathcal{A}$ are defined in the following way.

$$x * y = \begin{cases} 0 & \text{if } x \leq n(y), \\ \min\{x, y\} & \text{otherwise.} \end{cases} \qquad x \to y = \begin{cases} 1 & \text{if } x \leq y, \\ \max\{n(x), y\} & \text{otherwise.} \end{cases}$$

Where $n : A \to A$ is a negation function, i.e. $n(1) = 0$, $n(n(x)) \geq x$, and if $x < y$, then $n(x) \geq n(y)$. A negation fixpoint is an element x such that $n(x) = x$. Observe that $n(x) = \neg x$, for every $x \in A$.

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The variety of WNM-algebras is denoted by $\mathbb{WNM}$. Every subdirectly irreducible WNM-algebra is totally ordered: as a consequence, every subvariety of WNM-algebras is generated by its chains.

Examples: G, DP and NM-chains

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$$n(x) = \begin{cases} 0 & \text{if } x = 1, \\ 1 & \text{if } x = 0, \\ f & \text{otherwise.} \end{cases} \qquad x * y = \begin{cases} 0 & \text{if } x, y < 1, \\ \min\{x, y\} & \text{otherwise.} \end{cases}$$

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In every NM-chain the negation n is involutive, i.e. $n(n(x)) = x$.

Definition

Let $\mathcal{A}$ be a WNM-chain. Let us define the following sets.

- $S(A) = \{a \in A \mid \neg a = 0, a \neq 1\}$.
- $F(A) = \{a \in A \mid \neg a = \neg\neg a\}$.
- $I^-(A) = \{a \in A \mid \neg\neg a = a, 0 < a < \neg a\}$.
- $I^+(A) = \{a \in A \mid \neg\neg a = a, 1 > a > \neg a\}$.
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Negations of WNM-chains and some examples

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Theorem

A WNM-chain $\mathcal{A}$, with $|A| > 2$ is:

- *A Gödel-chain iff $A = S(A) \cup \{0, 1\}$, i.e. all the elements have a strict negation.*
- *A DP-chain iff $A = F(A) \cup \{0, 1\}$, i.e. A has a negation fixpoint f, and $\neg a = f$, for every $a \in A \setminus \{0, 1\}$.*
- *An NM-chain iff $A = I(A) \cup F(A)$, and $|F(A)| \leq 1$, i.e. every element has an involutive negation, and there is possibly a negation fixpoint.*

DNMG-algebras: first results

$\mathbb{DNMG}$ is the variety of WNM-algebras satisfying the following identity.

(DNMG) $$\neg\neg x \vee (\neg\neg x \to x) \vee (\neg\neg x \leftrightarrow \neg x) = 1.$$

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Theorem

A chain $\mathcal{A}$ is generic for $\mathbb{DNMG}$ iff the sets $F(A), I(A)$ and $S(A)$ are infinite.

The lattice of subvarieties of $\mathbb{DNMG}$, some ingredients

Let us consider the ordinal $\omega + 1$, i.e. $\mathbb{N}$ with the usual order plus a top element ω.

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With $(\omega + 1)^3$ we denote the lattice obtained as a direct product of three copies of $\omega + 1$, with the pointwise order: i.e. $(a, b, c) \leq (d, e, f)$ iff $a \leq d, b \leq e, c \leq f$.

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- *The poset $(\omega + 1)^3$ is denumerable.*
- *Every antichain in $(\omega + 1)^3$ is finite.*
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Our aim is to show that the lattice of subvarieties of $\mathbb{DNMG}$ (ordered by inclusion), $\Lambda(\mathbb{DNMG})$ is isomorphic to $\mathcal{AC}(\omega + 1)^3)$, that is $AC(\omega + 1)^3)$ equipped with an order such that $X \leq Y$ iff for every triple $x \in X$ there is $y \in Y$ such that $x \leq y$.

Definition

Let $\mathcal{A}$ be a DNMG-chain. Then the *triplet* $T(\mathcal{A})$ associated with $\mathcal{A}$ is an element $(a, b, c) \in (\omega + 1)^{(3)}$ defined as follows.

1. If $S(A)$ is infinite then $a = \omega$, otherwise $a = |S(A)|$.
2. If $I^-(A)$ is infinite then $b = \omega$, otherwise $b = |I^-(A)|$.
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Lemma

Let $\mathcal{A}$ and $\mathcal{B}$ be two DNMG-chains. Then,

- $\mathbf{V}(\mathcal{A}) \subset \mathbf{V}(\mathcal{B})$ *iff* $T(\mathcal{A}) < T(\mathcal{B})$.
- $\mathbf{V}(\mathcal{A}) = \mathbf{V}(\mathcal{B})$ *iff* $T(\mathcal{A}) = T(\mathcal{B})$.

DNMG-chains as triplets

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Some examples.

- If $\mathcal{A}$ is a Gödel-chain, then $T(\mathcal{A})$ has the form $(a, 0, 0)$.
- If $\mathcal{A}$ is a DP-chain, then $T(\mathcal{A})$ has the form $(0, 0, a)$.
- If $\mathcal{A}$ is an NM-chain, then $T(\mathcal{A})$ has the form $(0, a, b)$, with $b \leq 1$.
- If $\mathcal{A}$ is generic DNMG-chain, then $T(\mathcal{A}) = (\omega, \omega, \omega)$.

Theorem

The lattice $\Lambda(\mathbb{DNMG})$ *is isomorphic with* $\mathcal{AC}((\omega+1)^{(3)})$.

The structure of the lattice of subvarieties of $\mathbb{DNMG}$

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Corollary

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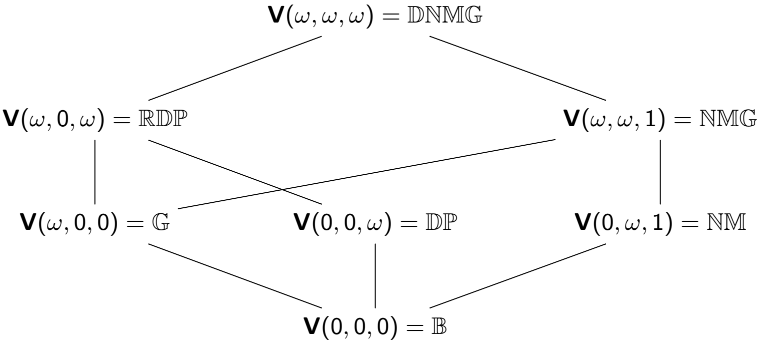
We also have a complete axiomatization for every variety in $\Lambda(\mathbb{DNMG})$

Theorem

$\mathbb{DNMG}$ *is the smallest subvariety in* $\Lambda(\mathbb{DNMG})$ *which contains* $\mathbb{DP}$, $\mathbb{NM}$, $\mathbb{G}$ *and it is generated by a single chain. In particular,* $\mathbb{DNMG}$ *is also generated by a standard DNMG-chain.*

The structure of $\Lambda(\mathbb{DNMG})$, some varieties



- W. Blok and D. Pigozzi, *Algebraizable logics*, vol. 77, Memoirs of The American Mathematical Society, no. 396, American Mathematical Society, 1989.
- F. Esteva and L. Godo, *Monoidal t-norm based logic: Towards a logic for left-continuous t-norms*, Fuzzy Sets Syst. **124** (2001), no. 3, 271–288.
- J. Fodor, *Nilpotent minimum and related connectives for fuzzy logic*, Proceedings of 1995 IEEE International Conference on Fuzzy Systems, IEEE, 1995, pp. 2077–2082.

APPENDIX

MTL-algebras

An MTL-algebra is an algebra $\langle A, *, \to, \wedge, \vee, 0, 1\rangle$. such that:

1. $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded lattice with minimum 0 and maximum 1.
2. $\langle A, *, 1\rangle$ is a commutative monoid.
3. $\langle *, \to\rangle$ forms a *residuated pair*: $z * x \leq y$ iff $z \leq x \to y$ for all $x, y, z \in A$. In particular, it holds that $x \to y = \max\{z \in A : z * x \leq y\}$.
4. The following equation holds.

   (Prelinearity) $$(x \to y) \vee (y \to x) = 1.$$

   A totally ordered MTL-algebra is called *MTL-chain*. An MTL-algebra is standard whenever it has $\langle [0, 1], \min, \max, 0, 1\rangle$ as lattice reduct.

- The class of MTL-algebras forms a variety, called $\mathbb{MTL}$. The logic corresponding to MTL-algebras is called MTL.
- An axiomatic extension of MTL is a logic obtained by adding other axioms to it.
- Every axiomatic extension of MTL is algebraizable in the sense of [BP89], and hence every subvariety of $\mathbb{MTL}$ induces a logic.

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Axiomatization of MTL

The basic connective are $\{\wedge, \&, \to, \bot\}$ (formulas built inductively: a theory is a set of formulas). Useful derived connectives are the following ones:

(negation)	$\neg\varphi \overset{\text{def}}{=} \varphi \to \bot$
(disjunction)	$\varphi \vee \psi \overset{\text{def}}{=} ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$
(top)	$\top \overset{\text{def}}{=} \neg\bot$

MTL can be axiomatized by using these axioms and modus ponens: $\frac{\varphi \quad \varphi\to\psi}{\psi}$.

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$(\varphi \& \psi) \to \varphi$
(A3)	$(\varphi \& \psi) \to (\psi \& \varphi)$
(A4)	$(\varphi \wedge \psi) \to \varphi$
(A5)	$(\varphi \wedge \psi) \to (\psi \wedge \varphi)$
(A6)	$(\varphi \& (\varphi \to \psi)) \to (\psi \wedge \varphi)$
(A7a)	$(\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi)$
(A7b)	$((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
(A8)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A9)	$\bot \to \varphi$

back

Axiomatization of the varieties in $\Lambda(\mathbb{DNMG})$

Theorem

For each $(a, b, c) \in (\omega + 1)^{(3)}$, *the variety* $\mathbf{V}(a, b, c)$ *is the subvariety of* $\mathbb{DNMG}$ *satisfying the following* ▸ *equation*.

$$(Q_a \sqcup e_{a+1}(S)) \sqcap (Q_b \sqcup e_{b+1}(I)) \sqcap (Q_c \sqcup e_{c+1}(F)) = 1\,.$$

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Theorem

Let $C = \{\mathcal{A}_i\}_{i \in I}$ be an irredundant set of DNMG-chains. Let further $t_i(x_1, \ldots, x_{n_i}) = 1$ be the equation axiomatising $\mathbf{V}(\mathcal{A}_i)$ for each $i \in I$, as given by the previous theorem. Then $\mathbf{V}(C)$ contains exactly the DNMG-algebras satisfying the equation

$$\bigsqcup_{i \in I} t_i(y_{i,1}, \ldots, y_{i,n_i}) = 1\,,$$

where all the variables $y_{i,j}$, for $i \in I$, and $j \in \{1, \ldots, n_i\}$, are pairwise distinct.

back

Some ancillary terms and equations

Definition

For each $n > 0$, we let $e_n(F)$, $e_n(I)$, $e_n(S)$ and Q_n denote the following terms:

$$e_n(F) = \bigsqcup_{i=1}^{n} ((\sim\sim x_i) \sqcup ((\sim\sim x_i \Rightarrow x_i) \sqcap (\sim((\sim(x_i^2))^2) \Leftrightarrow (\sim((\sim x_i)^2)))^2),$$

$$e_n(I) = \bigsqcup_{i=1}^{n} ((\sim\sim x_i) \sqcup (\sim x_i) \sqcup (\sim\sim x_i \Leftrightarrow \sim x_i)),$$

$$e_n(S) = \bigsqcup_{i=1}^{n} ((\sim\sim x_i \Leftrightarrow \sim x_i) \sqcup (\sim\sim x_i \Rightarrow x_i)),$$

$$Q_n = \bigsqcup_{1 \leq i \neq j \leq n+1} (x_i \Leftrightarrow x_j).$$

Furthermore let $Q_0 = 0$, $Q_\omega = 1$, and $e_{\omega+1}(X) = 1$ for each $X \in \{F, I, S\}$.

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