Logics of Variable Inclusion

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Contents

Matrices

Definition

1. A (logical) matrix is a pair $\langle \boldsymbol{A}, F \rangle$ where $\boldsymbol{A}$ is an algebra and $F \subseteq A$ (F is called the filter of the matrix).
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$$\Gamma \vdash_{\mathsf{M}} \varphi \iff \text{for every } \langle \boldsymbol{A}, F \rangle \in \mathsf{M} \text{ and hom } v \colon \boldsymbol{Fm} \to \boldsymbol{A}$$
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Matrices as models of logics

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Let $\vdash$ be a logic.

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- $\vdash$ is complete with respect to a class of matrices M when $\vdash = \vdash_{\mathsf{M}}$.
- We set $\mathrm{Mod}(\vdash) \coloneqq \{\langle \boldsymbol{A}, F \rangle : \langle \boldsymbol{A}, F \rangle \text{ is a model of } \vdash\}$.

Leibniz and Suszko congruence

- We need a process to identify the “important” matrices in $\mathrm{Mod}(\vdash)$:

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3. The Suszko congruence of F (over $\boldsymbol{A}$) is $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}F := \{\bigcap \boldsymbol{\Omega}G : G \supseteq F \text{ and } G \text{ is a filter}\}$

Reduced models

Definition

Let $\vdash$ be a logic.

1. The class of Leibniz reduced models of $\vdash$ is

$$\mathrm{Mod}^{*}(\vdash) := \{\langle \boldsymbol{A}, F\rangle \in \mathrm{Mod}(\vdash) : \boldsymbol{\Omega}^{\boldsymbol{A}} F = \mathrm{Id}_{\boldsymbol{A}}\}$$

2. The class of Suszko reduced models of $\vdash$ is

$$\mathrm{Mod}^{\mathrm{Su}}(\vdash) := \{\langle \boldsymbol{A}, F\rangle \in \mathrm{Mod}(\vdash) : \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}} F = \mathrm{Id}_{\boldsymbol{A}}\}$$

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- In most cases, reduced models (as opposed to arbitrary models) of a logic are its intended matrix semantics.

Direct systems of algebras

Definition

A direct system of algebras consists in

1. A semilattice $\boldsymbol{I} = \langle I, \vee \rangle$;
2. An indexed family of algebras $\{\boldsymbol{A}_i : i \in I\}$ with disjoint universes;
3. A homomorphism $f_{ij} \colon \boldsymbol{A}_i \to \boldsymbol{A}_j$, for every $i, j \in I$ such that $i \leq j$

such that f_{ii} is the identity map for every $i \in I$, and if $i \leq j \leq k$, then $f_{ik} = f_{jk} \circ f_{ij}$.

Płonka sums over a direct system of algebras

Definition

Let X be a direct system of algebras. The *Płonka sum* over X is a new algebra $\mathcal{P}_I(X)$ s.t.

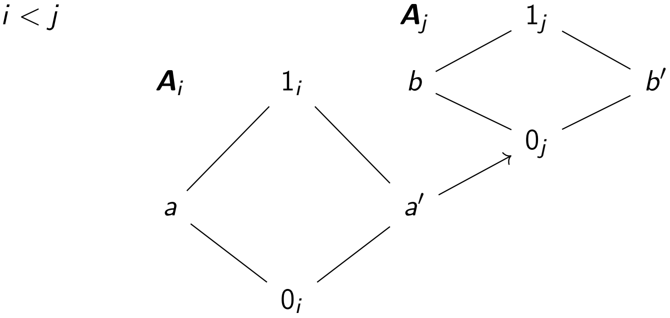
1. the universe of $\mathcal{P}_I(X) = \bigcup_{i \in I} A_i$
2. for every n-ary basic operation f on $\boldsymbol{A}_i$ and $a_1, \ldots, a_n \in \bigcup_{i \in I} A_i$, we set

$$f^{\mathcal{P}_I(\boldsymbol{A}_i)_{i \in I}}(a_1, \ldots, a_n) := f^{\boldsymbol{A}_j}(f_{i_1 j}(a_1), \ldots, f_{i_n j}(a_n))$$

where $a_1 \in A_{i_1}, \ldots, a_n \in A_{i_n}$ and $j = i_1 \vee \cdots \vee i_n$.

Płonka sums: example

Consider the following semilattice direct system of Boolean algebras



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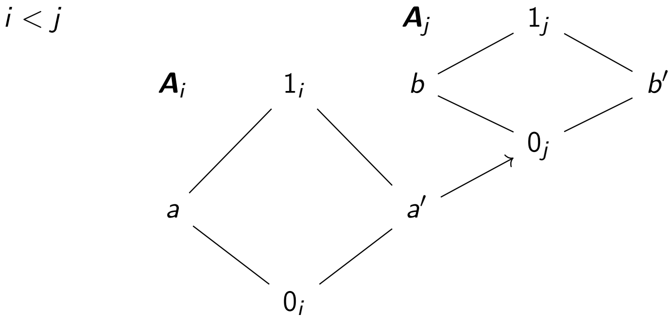

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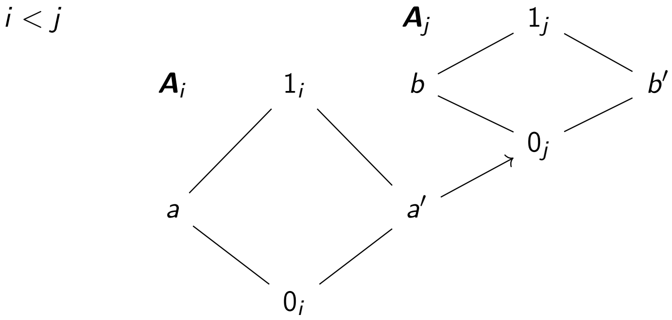

- $a \wedge^{\mathcal{P}_I} a' = a \wedge^{\boldsymbol{A}_i} a' = 0_i$,
- $a' \wedge^{\mathcal{P}_I} b = f_{ij}(a') \wedge^{\boldsymbol{A}_j} b = 0_j \wedge^{\boldsymbol{A}_j} b = 0_j$.

Płonka sums over a direct system of matrices

We extend the previous definitions to logical matrices as follows.

Definition

A *direct system* of matrices consists in

1. A join-semilattice $I = \langle I, \vee \rangle$.
2. A family of matrices $\{\langle \boldsymbol{A}_i, F_i \rangle : i \in I\}$.
3. A homomorphism $f_{ij} \colon \boldsymbol{A}_i \to \boldsymbol{A}_j$ such that $f_{ij}[F_i] \subseteq F_j$, for every $i, j \in I$ such that $i \leq j$.

Given directed system of matrices X as above, we set

$$\mathcal{P}_I(X) := \langle \mathcal{P}_I(\boldsymbol{A}_i)_{i \in I}, \bigcup_{i \in I} F_i \rangle.$$

The matrix $\mathcal{P}_I(X)$ is the *Płonka sum* of the direct system of matrices X. Given a class M of matrices, we denote by $\mathcal{P}_I(\mathrm{M})$ the class of all Płonka sums of directed systems of matrices in M.

Contents

Definition and examples

Definition

Let $\vdash$ be a logic. The logic of variable inclusion of $\vdash$ (or its *regularization*) is the relation $\vdash^r \subseteq \mathcal{P}(Fm) \times Fm$ defined as follows:

$$\Gamma \vdash^r \varphi \iff \text{there is } \Gamma' \subseteq \Gamma \text{ s.t. } Var(\Gamma') \subseteq Var(\varphi) \text{ and } \Gamma' \vdash \varphi.$$

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Example

Let $\vdash_{\mathcal{CL}}$ be classical logic. Its logic of variable inclusion $\vdash^r_{\mathcal{CL}}$ is the logic $\vdash_{\mathcal{PWK}}$ known as Paraconsistent Weak Kleene logic.

Models

Lemma (Soundness)

Let $\vdash$ be a logic and X be a direct systems of models of $\vdash$. Then $\mathcal{P}_I(X)$ is a model of $\vdash^r$.

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Let $\vdash$ *be a logic and* X *be a direct systems of models of* $\vdash$*. Then* $\mathcal{P}_I(X)$ *is a model of* $\vdash^r$*.*

Theorem (Completeness)

Let $\vdash$ *be a logic and* M *be a class of matrices containing the matrix* $\langle \mathbf{1}, \{1\} \rangle$*. If* $\vdash$ *is complete w.r.t.* M*, then* $\vdash^r$ *is complete w.r.t.* $\mathcal{P}_I(\mathsf{M})$*.*

Models

Lemma (Soundness)

Let $\vdash$ be a logic and X be a direct systems of models of $\vdash$. Then $\mathcal{P}_l(X)$ is a model of $\vdash^r$.

Theorem (Completeness)

Let $\vdash$ be a logic and M be a class of matrices containing the matrix $\langle \mathbf{1}, \{1\} \rangle$. If $\vdash$ is complete w.r.t. M, then $\vdash^r$ is complete w.r.t. $\mathcal{P}_l(\mathsf{M})$.

Corollary

Let $\vdash$ be a logic. Its logic of variable inclusion $\vdash^r$ is complete w.r.t. any of the following classes of matrices:

$$\mathcal{P}_l(\mathsf{Mod}(\vdash)) \quad \mathcal{P}_l(\mathsf{Mod}^*(\vdash)) \quad \mathcal{P}_l(\mathsf{Mod}^{Su}(\vdash)).$$

Logics with a partition function

Definition (Essentially Płonka)

A logic $\vdash$ has a partition function if there is a formula $x \cdot y$ in which the variables x and y really occur such that $x \vdash x \cdot y$ and the following equations hold in $\{\boldsymbol{A} : \exists F \subseteq A \text{ s.t. } \langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash)\}$ for every n-ary connective f:

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1. $a \cdot a = a$
2. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
3. $a \cdot (b \cdot c) = a \cdot (c \cdot b)$
4. $f(a_1, \dots, a_n) \cdot b = f(a_1 \cdot b, \dots, a_n \cdot b)$
5. $b \cdot f(a_1, \dots, a_n) = b \cdot a_1 \cdot_{\dots} \cdot a_n$

Note that in every logic with a lattice reduct the term $x \cdot y = x \wedge (x \vee y)$ is a partition function!

Suszko reduced models

Theorem

Let $\vdash$ *be a logic with a partition function* $\cdot$, *and let* X *be a directed system of matrices in* $\mathrm{Mod}^{Su}(\vdash)$. *TFAE:*

1. $\mathcal{P}_I(X) \in \mathrm{Mod}^{Su}(\vdash^r)$.
2. *For every* $n, i \in I$ *such that* $\langle \boldsymbol{A}_n, F_n \rangle$ *is trivial and* $n < i$, *there exists* $j \in I$ *s.t.* $n \leq j, i \nleq j$ *and* $\boldsymbol{A}_j$ *is non-trivial.*

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The theorem identifies the Suszko reduced models of $\vdash^r$, which can be expressed in terms of Płonka sums of Suszko reduced models of $\vdash$. Is it true that all Suszko models of $\vdash^r$ are of this kind? In general, the answer is no.

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However....

Suszko reduced models for a logic with inconsistency terms

- Refined characterization of $\mathrm{Mod}^{\mathsf{Su}}(\vdash^r)$ for $\vdash$ possessing a set of inconsistency terms;
- Full characterisation of $\mathrm{Mod}^{\mathsf{Su}}(\vdash^r)$ for $\vdash$ finitary and equivalential

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Hilbert-style axiomatisation

Definition

Let $\mathcal{H}$ be a Hilbert-style calculus with finite rules, which determines a logic $\vdash$ with a partition function $\cdot$. Let $\mathcal{H}^r$ be the Hilbert-style calculus given by the following rules:

$$\emptyset \rhd \psi \tag{1}$$

$$\gamma_1, \dots, \gamma_n \rhd \varphi \cdot \gamma_1 \cdot \gamma_2 \cdot \dots \cdot \gamma_{n-1} \cdot \gamma_n \tag{2}$$

$$\gamma \rhd \gamma \cdot \varphi \tag{3}$$

$$\chi(\epsilon, \vec{z}) \lhd \rhd \chi(\delta, \vec{z}) \tag{4}$$

for every

1. $\emptyset \rhd \psi$ rule in $\mathcal{H}$;
2. $\gamma_1, \dots, \gamma_n \rhd \varphi$ rule in $\mathcal{H}$;
3. $\epsilon \approx \delta$ equation in the definition of partition function, and formula $\chi(v, \vec{z})$.

Hilbert-style axiomatization

Theorem

Let $\vdash$ be a logic defined by a Hilbert-style calculus with finite rules $\mathcal{H}$. Then $\mathcal{H}^r$ is a complete Hilbert-style calculus for $\vdash^r$.

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Example

Hilbert-calculus for $\vdash_{\mathsf{CL}}$:

1. $\rhd x \to (y \to x)$
2. $\rhd x \to (y \to z) \to ((x \to y) \to (x \to z))$
3. $\rhd (x \to y) \to (\neg y \to \neg x)$
4. $x, x \to y \rhd y$

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3. $\rhd (x \to y) \to (\neg y \to \neg x)$
4. $x, x \to y \rhd y$

Hilbert-calculus for $\vdash^r_{\mathsf{CL}} = \vdash_{\mathsf{PWK}}$:

- (1-3) as axioms
- $x, x \to y \rhd y \wedge (y \vee (x \wedge (x \vee x \to y)))$
- $x \rhd x \wedge (x \vee y)$
- $\chi(\delta, \vec{z}) \lhd\rhd \chi(\epsilon, \vec{z})$.

Contents

Further results

- Classification of $\vdash^r$ within the Leibniz Hierarchy

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- Algebraizability of Gentzen systems associated with $\vdash^r$

Thank you!