Arnon Avron
School of Computer Science
Tel Aviv University

Purely Relevant Logics with Contraction and Its Converse

Second SYSMICS Workshop
Vienna, 2018

What is a Substructural Propositional Logic?

- Each connective of the language has a classical counterpart.
- The consequence relation can be defined using a Gentzen-type system having two types of rules:

  Structural Rules: Do not involve any specific connective

  Logical Rules: Each involving exactly one connective.
- The set of rules for each of the connectives is adequate for its classical counterpart.

Some Structural Rules

Weakening:

$$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$$

Contraction:

$$\frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}$$

Expansion:

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi, \varphi, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi, \varphi}$$

Mingle:

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \quad \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Expansion=Mingle?

- Mingle+Contraction entail Expansion:

$$\dfrac{\dfrac{\varphi, \Gamma \Rightarrow \Delta \quad \varphi, \Gamma \Rightarrow \Delta}{\varphi, \varphi, \Gamma, \Gamma \Rightarrow \Delta, \Delta}\ (Mingle)}{\varphi, \varphi, \Gamma \Rightarrow \Delta}\ (Contractions)$$

- Expansion+$\mathbf{1}$ entail Mingle:

$$\dfrac{\Rightarrow \mathbf{1} \qquad \dfrac{\dfrac{\dfrac{\mathbf{1} \Rightarrow \mathbf{1}}{\mathbf{1} \Rightarrow \mathbf{1}, \mathbf{1}}\ (Exp) \quad \dfrac{\Gamma_1 \Rightarrow \Delta_1}{\mathbf{1}, \Gamma_1 \Rightarrow \Delta_1}\ (\mathbf{1} \Rightarrow)}{\mathbf{1}, \Gamma_1 \Rightarrow \Delta_1, \mathbf{1}}\ (Cut) \quad \dfrac{\Gamma_2 \Rightarrow \Delta_2}{\mathbf{1}, \Gamma_2 \Rightarrow \Delta_2}\ (\mathbf{1} \Rightarrow)}{\mathbf{1}, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (Cut)}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (Cut)$$

Blame it on $\&$

- Expansion+$\{\rightarrow, \&\}$ entail Mingle:

  Replace $\mathbf{1}$ by a sentence of the form

$$(\varphi_1 \rightarrow \varphi_1)\& \ldots \&(\varphi_n \rightarrow \varphi_n)$$

- Expansion+$\&$ suffice for deriving $\Rightarrow \varphi, \psi$ from $\Rightarrow \varphi$ and $\Rightarrow \psi$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow \varphi \quad \Rightarrow \psi}{\Rightarrow \varphi\&\psi}}{\Rightarrow \varphi\&\psi, \varphi\&\psi} \quad \dfrac{\varphi \Rightarrow \varphi}{\varphi\&\psi \Rightarrow \varphi}}{\Rightarrow \varphi, \varphi\&\psi} \quad \dfrac{\psi \Rightarrow \psi}{\varphi\&\psi \Rightarrow \psi}}{\Rightarrow \varphi, \psi}$$

The Substructural System RMI_m

Structural Rules: Contraction, Expansion

Logical Rules:

$$(\neg\Rightarrow)\quad \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta} \qquad\qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}\quad (\Rightarrow\neg)$$

$$(\otimes\Rightarrow)\quad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \otimes \psi \Rightarrow \Delta} \qquad\qquad \frac{\Gamma_1 \Rightarrow \Delta_1, \varphi \qquad \Gamma_2 \Rightarrow \Delta_2, \psi}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \varphi \otimes \psi}\quad (\Rightarrow\otimes)$$

$$(\rightarrow\Rightarrow)\quad \frac{\Gamma_1 \Rightarrow \Delta_1, \varphi \qquad \psi, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \varphi \rightarrow \psi \Rightarrow \Delta_1, \Delta_2} \qquad\qquad \frac{\Gamma, \varphi \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi}\quad (\Rightarrow\rightarrow)$$

Properties of RMI_m

- An equivalent Hilbert-type system $HRMI_m$ is obtained by adding to R_m the mingle axiom $\varphi \otimes \varphi \to \varphi$:
  1. A sequent is provable in RMI_m iff its translation is provable in $HRMI_m$. In particular: $\vdash_{RMI_m} \Rightarrow \varphi$ iff $\vdash_{HRMI_m} \varphi$.
  2. $\mathcal{T} \vdash_{HRMI_m} \varphi$ iff $\vdash_{RMI_m} \Gamma \Rightarrow \varphi$ for some finite $\Gamma \subseteq \mathcal{T}$.
- Relevant Deduction Theorem: $\mathcal{T}, \varphi \vdash_{RMI_m} \psi$ iff either $\mathcal{T} \vdash_{RMI_m} \psi$ or $\mathcal{T} \vdash_{RMI_m} \varphi \to \psi$.
- Variable-sharing: If $\vdash_{RMI_m} \varphi \to \psi$ then φ and ψ share a variable.
- If $\vdash_{RMI_m} \varphi \otimes \psi$ then $\vdash_{RMI_m} \varphi$ and $\vdash_{RMI_m} \psi$.

Weakly Characteristic Semantics

The structure $\mathcal{A}_\omega = \langle A_\omega, \mathcal{D}_\omega, \mathcal{O}_\omega \rangle$ is defined as follows:

- $A_\omega = \{\mathbf{t}, \mathbf{f}, I_1, I_2, I_3, \ldots\}$
- $\mathcal{D}_\omega = A_\omega - \{\mathbf{f}\} = \{\mathbf{t}, I_1, I_2, I_3, \ldots\}$.
- The operations in $\mathcal{O}_\omega$ are the following:

$$\neg\mathbf{t} = \mathbf{f} \quad \neg\mathbf{f} = \mathbf{t} \quad \neg I_k = I_k \quad (k = 1, 2, \ldots)$$

$$a \otimes b = \begin{cases} \mathbf{f} & a = \mathbf{f} \text{ or } b = \mathbf{f} \\ I_k & a = b = I_k \\ \mathbf{t} & otherwise \end{cases} \qquad a \rightarrow b = \begin{cases} \mathbf{t} & a = \mathbf{f} \text{ or } b = \mathbf{t} \\ I_k & a = b = I_k \\ \mathbf{f} & otherwise \end{cases}$$

Weakly Characteristic Semantics (Continued)

Weak soundness and completeness: $\vdash_{RMI_m} \varphi$ iff $\vdash_{\mathcal{A}_\omega} \varphi$.

Corollary: $\vdash_{RMI_m} \Gamma \Rightarrow \Delta$ iff for every valuation v in $\mathcal{A}_\omega$, either $v(\varphi) = \mathbf{f}$ for some $\varphi \in \Gamma$, or $v(\varphi) = \mathbf{t}$ for some $\varphi \in \Delta$, or there exists k such that $v(\varphi) = I_k$ for every $\varphi \in \Gamma \cup \Delta$.

Scroggs' property: RMI_m does not have a finite (weakly) characteristic matrix, although every proper extension of it does.

Strongly Characteristic Semantics

- $SA = [0,1] \times A_\omega$

  If $v = \langle x, a \rangle \in SA$ then $deg(v) = x$ and $val(v) = a$.

- $\mathcal{D} = [0,1] \times \mathcal{D}_\omega$ (where $\mathcal{D}_\omega = \{\mathbf{t}, I_1, I_2, I_3, \ldots\}$).

- $\neg\langle x, \mathbf{t} \rangle = \langle x, \mathbf{f} \rangle \quad \neg\langle x, \mathbf{f} \rangle = \langle x, \mathbf{t} \rangle \quad \neg\langle x, I_k \rangle = \langle x, I_k \rangle$

$$deg(u \otimes v) = min\{deg(u), deg(v)\}$$

$$val(u \otimes v) = \begin{cases} I_k & u = v \text{ and } val(u) = I_k \\ \mathbf{f} & deg(u) \leq deg(v) \text{ and } val(u) = \mathbf{f} \\ \mathbf{f} & deg(u) \geq deg(v) \text{ and } val(v) = \mathbf{f} \\ \mathbf{t} & \text{otherwise} \end{cases}$$

We denote the resulting structure by $\mathcal{SA}$.

Strongly Characteristic Semantics (Continued)

- Strong soundness and completeness: $\mathcal{T} \vdash_{RMI_m} \varphi$ iff $\mathcal{T} \vdash_{\mathcal{SA}} \varphi$.
- Suppose that $\Gamma \nvdash_{RMI_m} \varphi$, and that $\Gamma \cup \{\varphi\}$ involves at most n different propositional variables. Then there is a submatrix $\mathcal{SA}(\Gamma, \varphi)$ of $\mathcal{SA}$ such that $\mathcal{SA}(\Gamma, \varphi)$ has at most $3n - 1$ elements, and there is a valuation in it which is a model of Γ, but not a model of φ.
- For $n > 0$ there is a theory T_n in $p_1, \ldots, p_n$ such that $T_n \nvdash_{RMI_m} p_1$, but any model of T_n in $\mathcal{SA}$ which is not a model of p_1 involves at least n different degrees, and at least $3n - 1$ different elements of SA.

Related Partial Orders

On $\mathcal{A}_\omega$: $a \preceq^1 b$ if either $a = b$ or $a = \mathbf{f}$ or $b = \mathbf{t}$ ($\mathbf{f} \preceq^1 I_k \preceq^1 \mathbf{t}$).

On $\mathcal{SA}$**:**

- $\langle x, a \rangle \preceq_\otimes \langle y, b \rangle$ if either $x < y$ & $a \in \{\mathbf{t}, \mathbf{f}\}$, or $x = y$ and either $a = b$ or $a = \mathbf{f}$ or $a = \mathbf{t}$ & $b \neq \mathbf{f}$.

  $\langle x, \mathbf{f} \rangle \preceq_\otimes \langle x, \mathbf{t} \rangle \preceq_\otimes \langle x, I_k \rangle$, and if $x < y$ then $\langle x, \mathbf{t} \rangle \preceq_\otimes \langle y, \mathbf{f} \rangle$.

- $\langle x, a \rangle \preceq \langle y, b \rangle$ if either $x = y$ & $a \preceq^1 b$ or $x < y$ & $a = \mathbf{f}$, or $x > y$ & $b = \mathbf{t}$.

  If $x < y$ then $\langle x, \mathbf{f} \rangle \preceq \langle y, \mathbf{f} \rangle \preceq \langle y, I_k \rangle \preceq \langle y, \mathbf{t} \rangle \preceq \langle x, \mathbf{t} \rangle$.

Related Partial Orders (Continued)

- $\langle SA, \preceq_{\otimes} \rangle$ is a lower semilattice, and $u \otimes v = inf_{\preceq_{\otimes}}\{u, v\}$.
- $\langle SA, \preceq \rangle$ is a lattice.
- Denote the lattice operations for $\preceq$ by $\wedge$ and $\vee$. Then:
  - $u \preceq v$ iff $u \to v \in \mathcal{D}$.
  - $u \wedge v \preceq w$ iff $u \preceq v \to w$.
- However, $\mathcal{D}$ is not closed under $\wedge$!

Enriching the Language with "Additives"

- Add to the language the connectives $\wedge, \vee$ and the constants $\top, \bot$.
- Let RMI_{ma} be obtained from RMI_m by adding:

Axioms:

$$\bot, \Gamma \Rightarrow \Delta \qquad\qquad \Gamma \Rightarrow \Delta, \top$$

Rules:

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \qquad \frac{\psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi} \qquad \frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

Condition: in $(\Rightarrow \wedge)$ and $(\vee \Rightarrow)$ $\Gamma \cup \Delta$ should not be empty!

Enriching the Language with "Additives" (Continued)

- RMI_{ma} is a conservative extension of RMI_m.
- RMI_{ma} has the variable-sharing property.
- RMI_{ma} is sound, but not complete with respect to $\mathcal{SA}$. It is sound and complete w.r.t. a richer class of lattices.
- A cut-free formulation is obtained by replacing the Expansion rules of RMI_{ma} by the Relevant Mingle rules:

$$\frac{\varphi, \Gamma_1 \Rightarrow \Delta_1 \quad \varphi, \Gamma_2 \Rightarrow \Delta_2}{\varphi, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \qquad \frac{\Gamma_1 \Rightarrow \Delta_1, \varphi \quad \Gamma_2 \Rightarrow \Delta_2, \varphi}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \varphi}$$

Hypersequential Version of RMI_{ma}

Hypersequent: $s_1 \mid s_2 \mid \ldots \mid s_n$ where $s_1, \ldots, s_n$ are sequents.

- The hypersequential versions of the rules of RMI_{ma}. E.g.:

$$\frac{G \mid \Gamma, \varphi, \psi \Rightarrow \Delta}{G \mid \Gamma, \varphi \otimes \psi \Rightarrow \Delta} \qquad \frac{G_1 \mid \Gamma_1 \Rightarrow \Delta_1, \varphi \qquad G_2 \mid \Gamma_2 \Rightarrow \Delta_2, \psi}{G_1, G_2 \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \varphi \otimes \psi}$$

- The basic External structural rules:

$$\frac{G \mid s \mid s}{G \mid s} \qquad \frac{G}{G \mid s}$$

The Logics of $\mathcal{SA}$ and $\mathcal{A}_\omega$

RMI - **the logic of $\mathcal{SA}$:** Add to RMI_{ma} the Splitting rule:

$$\frac{G \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{G \mid \Gamma_1 \Rightarrow \Delta_1 | \Gamma_2 \Rightarrow \Delta_2}$$

$SRMI$ - **the logic of $\mathcal{A}_\omega$:** Add to RMI_{ma} Strong Splitting:

$$\frac{G \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{G \mid \Gamma_1 \Rightarrow \Delta_1 | \Gamma_2, \Gamma' \Rightarrow \Delta_2, \Delta'}$$

In both cases we have soundness, completeness and cut-elimination.