MIX $\star$-AUTONOMOUS QUANTALES
AND THE CONTINUOUS WEAK BRUHAT ORDER

MARIA JOÃO GOUVEIA AND LUIGI SANTOCANALE

We shall illustrate in this talk some applications of sub-structural logics within lattice theory.

Our previous researches focused on lattices of permutations—known as the Permutohedra or the weak Bruhat orders—and related constructions giving rise to exotic lattices [2]. Intriguing lattices generalizing Permutohedra are the multinomial lattices [1]. Its elements, words on a finite alphabet $\{ x, y, z \ldots \}$ with fixed number of occurrences of each letter, can be given a geometrical interpretation as discrete paths in some multidimensional Euclidean space. In our TACL 2011 talk [3] we presented the following result, showing that the lattice structure can be lifted from the discrete to the continuous case.

Proposition. *Let $d \geq 2$. Images of increasing continuous paths from $\vec{0}$ to $\vec{1}$ in $\mathbb{R}^d$ can be given the structure of a lattice; moreover, all the Pemutohedra and all the multinomial lattices can be embedded into one of these lattices.*

We called this lattice the *continuous weak Bruhat order*. The proof that such an object is a lattice was clumsy, due to complicated computations stemming from real analysis. We recently discovered a cleaner proof, that exhibits the construction of the continuous weak Bruhat order as an instance of a general construction that relies on ideas from substructural logics.

Let $\langle Q, 1, \otimes, ^\star \rangle$ be a $\star$-autonomous quantale (or residuated lattice), denote by 0 and $\oplus$ the dual monoidal operations. We do not assume that Q is commutative, but we assume it is cyclic, so $x^\star = x \backslash 0 = 0/x$. Moreover, we assume that Q satisfies the MIX rule $x \otimes y \leq x \oplus y$. Let $I_d := \{ (i,j) \mid 1 \leq i < j \leq d \}$ and consider the product Q^{I_d}. Say that $f \in Q^{I_d}$ is closed if $f_{i,j} \otimes f_{j,k} \leq f_{i,k}$, and say that it is open if $f_{i,k} \leq f_{i,j} \oplus f_{j,k}$; say that f is clopen if it is closed and open.

Proposition. *The set of clopen tuples of Q^{I_d}, with the pointwise ordering, is a lattice.*

In particular, when Q is the quantale of completely additive functions from the unit interval to itself, which is $\star$-autonomous and satisfies the MIX rule, this construction yields the continuous weak Bruhat order in dimension d. As its proof relies on purely algebraic methods, the Proposition allows to reduce the equational theories of these lattices to the equational theory of the quantales they are built from.

References

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[3] Luigi Santocanale. The continuous weak bruhat order. In *Proocedings of the conference Topology, Algebra, and Categories in Logic TACL 2011*, 2011. Available at `http://pageperso.lif.univ-mrs.fr/~luigi.santocanale/tacl2011/slides/55.pdf`.

Maria João Gouveia, CEMAT-CIÊNCIAS, Universidade de Lisboa, 1749-016, Lisboa, Portugal
E-mail address: `mjgouveia@fc.ul.pt`

Luigi Santocanale, LIF, CNRS UMR 7279, Aix-Marseille Université
E-mail address: `luigi.santocanale@lif.univ-mrs.fr`