Abstract: Structural rules for multi-valued logics

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1 Introduction

Usually, under the term *structural rules*, one refers to the following rules, devised by Gentzen for incorporation in his sequent calculi LJ/LK for intuitionistic/classical logics.

$$\frac{\Gamma_1, \psi, \varphi, \Gamma_2 \rightarrow \chi}{\Gamma_1, \varphi, \psi, \Gamma_2 \rightarrow \chi}\ (PL) \qquad \frac{\Gamma \rightarrow \chi}{\Gamma, \varphi \rightarrow \chi}\ (WL) \qquad \frac{\Gamma, \varphi, \varphi \rightarrow \chi}{\Gamma, \varphi \rightarrow \chi}\ (CL) \tag{1.1}$$

Here Γ is a meta-variable over sequences of object language formulas, and φ, ψ and χ are meta-variables ranging over formulas. In the case of *multi-conclusions* sequents $\Gamma \rightarrow \Delta$, there are also analogous rules (PR), (WR), (CR) for modifying Δ, the r.h.s. of a sequent.

The characteristic property of those structural rules, inspiring their names, is that they do not refer to any *specific* connective/quantifier, in contrast to logical (or operational) rules, defining the meanings of the logical operators. However, their soundness does depend on the logics for which they are intended to be *bivalent*.

In this paper, we consider structural rules suitable for *multi-valued logics*, rules endowed with the same characteristic of not depending on the operators of the logic. Since the semantics is multi-valued, some syntactic means is needed for referring to the truth value of a formula, in particular within a derivation a proof-system. We propose such a generalization of a traditional sequent, and consider suitable structural rules applicable to this generalized structure.

2 Located formulas and sequents

Let $\mathcal{V} = \{v_1, \cdots, v_n\}$, $n \geq 2$ be the collection of truth values underlying a multi-valued logic L. Let $\hat{n} = \{1, \cdots, n\}$. A *located formula* (l-formula) is a pair (φ, k), where φ is a formula and $k \in \hat{n}$. The intended meaning of (φ, k) is the association of φ with the truth value $v_k \in \mathcal{V}$. A *located sequent* (l-sequent) has the form $\Pi = \Gamma \rightarrow \Delta$, whre Γ, Δ are (possibly empty) finite sets of l-formulas. We use $\mathbf{\Pi}$ for sets of l-sequents. Let σ range over truth value assignments, mapping formulas to truth values; for atomic sentences the mapping is arbitrary, and is extended to formulas so as to respect the truth tables of the operators. Below, we define the cental semantic notions as applicable to poly-sequents.

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Definition 2.1 (satisfaction, consequence)

satisfaction:

$$\sigma \models \Pi \text{ iff if } \sigma[\![\varphi]\!] = v_k \text{ for all } (\varphi, k) \in \Gamma, \text{ then } \sigma[\![\psi]\!] = v_j \text{ for some } (\psi, j) \in \Delta \tag{2.2}$$

consequence:

$$\mathbf{\Pi} \models \Pi \text{ iff } \sigma \models \Pi' \text{ for all } \Pi' \in \mathbf{\Pi} \text{ implies } \sigma \models \Pi \tag{2.3}$$

We are interested in proof-systems sound and complete for this consequence relation. The structural rules below are all sound for the multi-valued semantics. In [1, 2], logical rules for arbitrary logical operators are added, constructed from the truth tables in a uniform way. The whole system (and several of its variants) are shown to be also complete w.r.t. the multi-valued semantics. As we are interested here in structural rules only, we skip the discussion of logical rules.

3 Structural rules for l-sequents-based proof systems

The *initial l-sequents* are the natural generalization of Gentzen's bivalent identity rule

$$(\varphi, k) \rightarrow (\varphi, k), \text{ for each } k \in \hat{n} \tag{3.4}$$

The two central (families of) new structural rules, directly related to multi-valuedness, are the *switching rules*: For every $1 \leq i, j \leq n$:

$$\frac{\Gamma, (\varphi, i) \rightarrow \Delta}{\Gamma \rightarrow \Delta, \varphi \times \overline{\{i\}}} (\overrightarrow{s}_i) \qquad \frac{\Gamma \rightarrow \Delta, (\varphi, i)}{\Gamma, (\varphi, j) \rightarrow \Delta} (\overleftarrow{s}_{i,j}), j \neq i \tag{3.5}$$

To understand these rules, consider again Gentzen's rules for negation:

$$\frac{\Gamma \rightarrow \varphi, \Delta}{\Gamma, \neg\varphi \rightarrow \Delta} (\neg L) \qquad \frac{\Gamma, \varphi \rightarrow \Delta}{\Gamma \rightarrow \neg\varphi, \Delta} (\neg R) \tag{3.6}$$

The effect of these rules in classical, bivalent logic is that φ is false iff $\neg\varphi$ is true.

Suppose now that we want to avoid the use of negation, and, instead, refer directly to the falsity of a formula. Then, a reformulation of Gentzen's rule using l-sequents with $n = 2$, and using $t,\ f$ as mnemonic locating indices (instead of 1, 2), is as follows.

$$\begin{array}{cc} \dfrac{\Gamma \rightarrow (\varphi, t), \Delta}{\Gamma, (\varphi, f) \rightarrow \Delta} (\overleftarrow{s}_{t,f}) & \dfrac{\Gamma, (\varphi, t) \rightarrow \Delta}{\Gamma \rightarrow (\varphi, f), \Delta} (\overrightarrow{s}_t) \\ \dfrac{\Gamma \rightarrow (\varphi, f), \Delta}{\Gamma, (\varphi, t) \rightarrow \Delta} (\overleftarrow{s}_{f,t}) & \dfrac{\Gamma, (\varphi, f) \rightarrow \Delta}{\Gamma \rightarrow (\varphi, t), \Delta} (\overrightarrow{s}_f) \end{array} \tag{3.7}$$

In this case, a logical rule (about negation) is converted into structural rules (about truth and falsity). Here there is only one way to shift. Note that each negation rule breaks into two separate shifting rules, as truth and falsity need to be treated separately. The shifting rules above consider the general case, where not having some given truth value (out of n values) is related to having other truth values.

An l-sequent immediately derivable by shifting is $\rightarrow \{\varphi\} \times \hat{n}$:

$$\frac{(\varphi, k) \rightarrow (\varphi, k)}{\rightarrow \{\varphi\} \times \hat{n}} (\overrightarrow{s}_k) \tag{3.8}$$

(for an arbitrary $k \in \hat{n}$). This is a natural generalization of the *excluded middle*, the latter seen as stating that every φ is either true or false. Here, every φ has *some* truth value within $\mathcal{V}$.

The other central (family of) structural rules are the *coordination rules*:

$$\frac{\Gamma \rightarrow (\varphi, i), \Delta \quad \Gamma \rightarrow (\varphi, j), \Delta}{\Gamma \rightarrow \Delta}\ (c_{i,j}),\ \{i, j\} \subseteq \hat{n},\ i \neq j \tag{3.9}$$

These rules are a natural generalization of *ex contradictione (sequitur) quodlibet*, everything follows from a contradiction. A contradiction can be regarded as some φ being both true and false (and no need for negation!). Its generalization is some φ having two *different* truth values $v_i,\ v_j \in \mathcal{V},\ i \neq j$.

There remains an issue of sub-structurality regarding those multi-valued structural rules:

- The coordination rules $(c_{i,j})$ were formulated as *additive* (context sharing) rules. When is a multiplicative formulation (context free) inequivalent?
- What happens when either the switching rules or the coordination rules (or both) are *omitted*?

References

[1] Nissim Francez and Michael Kaminski. On poly-logistic natural-deduction. *Journal of Applied Logic (submitted)*, 2017. Presented at ISRALOG17, Haifa, October 2017.

[2] Michael Kaminski and Nissim Francez. Calculi for multi-valued logics. 2017.